M1410 Homework Worksheet:

1. Define two points $p, q \in \boldsymbol{R}^2$ to be equivalent if $p - q \in \boldsymbol{Z}^2$, the set of integer points. Let $\boldsymbol{T}^2 = \boldsymbol{R}^2/\boldsymbol{Z}^2$. Prove that $\boldsymbol{T}^2$ is a smooth manifold by finding coordinate charts to $\boldsymbol{R}^2$ such that the overlap functions are translations. $\boldsymbol{T}^2$ is the square torus.

2. Prove that there exists a closed 1-form on $\boldsymbol{T}^2$ which is not exact. This means that $H^1(\boldsymbol{T}^2)$ is nontrivial.

3. Let S^2 denote the 2-sphere, namely the solution to the equation

$$x^2 + y^2 + z^2 = 1$$

in $\boldsymbol{R}^3$. Let $\Delta \subset S^2$ be any round disk. Prove that there is a diffeomorphism from $S^2 - \Delta$ to an open disk in $\boldsymbol{R}^2$.

4. Suppose that ω is a closed 1-form on S^2. Let Δ be some disk in S^2 which does not contain $(0, 0, -1)$. Prove that there is a unique smooth function f on $S^2 - \Delta$ such that $f(0, 0, -1) = 0$ and $df = \omega$ on $S^2 - \Delta$. Hint: Problem 3 might be helpful.

5. Suppose that ω is a closed 1-form on S^2. Prove that there is some function f on all of S^2 such that $df = \omega$. Hint: Problem 4 might be helpful. Conclude that $H^1(S^2)$ is trivial.

6. Suppose that M_1 and M_2 are two compact manifolds and $F : M_1 \to M_2$ is a diffeomorphism. Prove that $H^k(M_1)$ and $H^k(M_2)$ are isomorphic vector spaces for any k. Deduce from this that there is no diffeomorphism from $\boldsymbol{T}^2$ to S^2. These are two compact 2-manifolds which are not diffeomorphic.

7. Generalize the argument above to show that the n-torus $\boldsymbol{T}^n$ is not diffeomorphic to the n-sphere S^n, for all $n > 2$.

8. (Bonus) Using H^1, show that the product manifold $(S^2 \times \boldsymbol{T}^2)$ is not diffeomorhic to S^4.