M1410 Homework Worksheet:

Most of this handout deals with a result related to the no-retraction theorem. Let S^n denote the n-dimensional unit sphere, given one of the two orienations. (It doesn't matter which one you pick, as long as you keep it the same throughout the handout.) Let B^{n+1} be the ball bounded by S^n.

Suppose that $f : S^n \to S^n$ is a smooth map. We say that f is *regular at* p if Df_p is nonsingular. Assuming that this is the case, define $s(f,p) = 1$ if f is locally orientation-preserving at p and -1 if f is locally orientation reversing at p.

Say that $q \in S^n$ is a *regular value* if $f^{-1}(q)$ consists of a finite number of points, say $p_1, ..., p_k$, and f is regular at each of these points. In this case, define

$$S(f,q) = \sum_{i=1}^{k} s(f,p_i).$$

The quantity $S(f,q)$ counts the number of pre-images of q, but with the orientation taken into account.

The goal of this worksheet is to guide you through the proof of the following theorem.

Theorem 0.1 *Suppose that $F : B^{n+1} \to B^{n+1}$ is smooth and $F(S^n) \subset S^n$. Let f be the restriction of F to S^n. Suppose that there is some $q \in S^n$ which is a regular value for f and $S(f,q) \neq 0$. Then F is surjective.*

Problem 1: Suppose that F satisfies all the hypotheses of the theorem but is not surjective. Prove that there exists a smooth map $G : B^{n+1} \to S^n$ such that $G(S^n) \subset S^n$ with the following additional property. Let g be the restriction of G to S^n. Then there is some $q \in S^n$ which is regular value of g such that $S(g,q) \neq 0$.

Problem 2: Let $\epsilon > 0$ be some arbitrary constant. Prove that there is a smooth form $\omega \in \Omega^n(S^n)$ whose support is contained in the ϵ ball about q such that

$$\int_{S^n} \omega = 1.$$

Problem 3: Let ω be the form from Problem 2. Prove that, if ϵ is small enough, then

$$\int_{S^n} G^*(\omega) = S(g, q).$$

Problem 4: Apply Stokes' Theorem to $G^*(\omega)$ to prove Theorem 0.1.

Problem 5: Prove, for any integer n, that there is a smooth map $f : S^2 \to S^2$ and a point $q \in S^2$ such that q is regular for f and $S(f, q) = n$.

Problem 6: Prove that there exists a smooth map $f : S^3 \to S^3$ together with a point $q \in S^3$ such that $S(f, q) = 2$.