Lemma 11.4 in the book

There seems to be a gap in the proof of Lemma 11.4 in the book. These notes explain the gap and fill it in. The gap isn't serious, but the proof definitely leaves a key point unexplained. One thing that tipped me off to the gap is that the proof never uses property (c) of Definition 11.2, and thus does not explain why it is needed. The result is false without property (c).

A Counterexample: Let $M \subset \boldsymbol{R}^2$ be the subset shown in Figure 1.

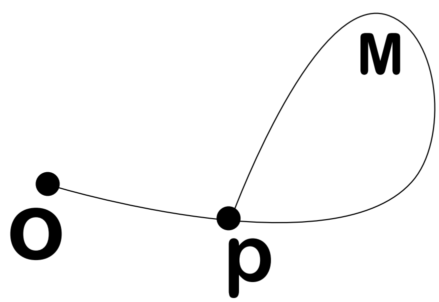

Figure 1: M is not a manifold around the point p.

There is a smooth injective map $\alpha : (0,2) \to M$ such that

- $\phi(0) = o$.
- $\phi(1) = p$.
- $\lim_{t\to 2} \phi(t) = p$

You can certainly make $d\phi$ nonsingular on $(0,2)$. If we take $V_\alpha = (0,2)$, then the map α is a coordinate chart in the sense of Definition 11.2, except that the inverse map α^{-1} is not continuous at p. The conclusion of Lemma 11.4 would be that α^{-1} is smooth on $V_\alpha = \alpha(U_\alpha) = M$, and this is false. α^{-1} is not even continuous.

A Near Miss: Now let's work with another example in which the given proof does not quite work. Figure 2 shows another manifold M. This time M really is a manifold – never mind the corners, they are just an illusion. This manifold has a (black) line segment Z which runs parallel to the (red) segment of M containing p. Let's call this black segment Z.

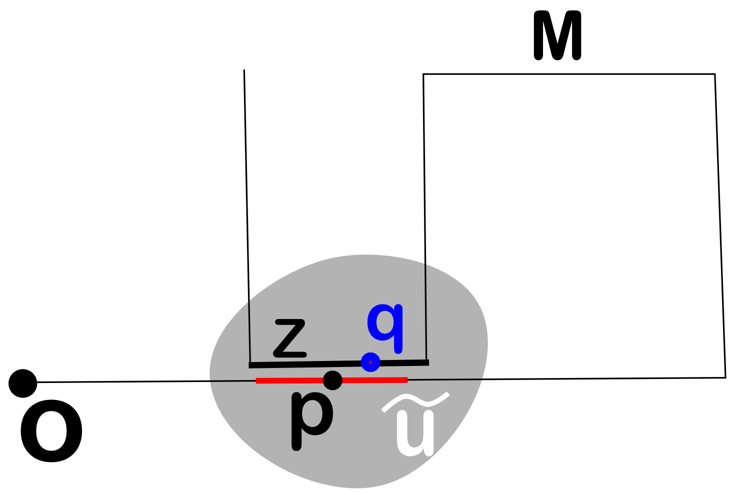

Figure 2: The set $\widetilde{U}$ intersects M in $G(\widetilde{V}) \cup Z$ instead of just $G(\widetilde{V})$.

The proof in Lemma 11.4 constructs open sets $\widetilde{V} \subset \boldsymbol{R}^k \times \boldsymbol{R}^{n-k}$ and $\widetilde{U} \subset \boldsymbol{R}^n$, and a diffeomorphism $G : \widetilde{V} \to \widetilde{U}$ with with $\widetilde{U} \cap M$ containing a neighborhood of p. Notive that nothing stops the neighborhood $\widetilde{U}$ from being large enough that it also contains the segment Z, even though Z does not lie in the image of G.

We have the map $F : \widetilde{U} \to \widetilde{V}$, where F is the inverse of G. The main goal of the proof is to show that the restriction of F to $\widetilde{U} \cap M$ is α^{-1}. Let $q \in \widetilde{U} \cap M$. The proof asserts that

$$q \in \alpha(V), \qquad \text{where} \qquad V = \widetilde{V} \cap (\boldsymbol{R}^k \times \{(0, ..., 0)\}). \tag{1}$$

However, this need not be the case: We might have $q \in Z$, as we have shown in Figure 2.

The way around the problem is to note that, **because α^{-1} is continuous**, there is a smaller neighborhood $\widetilde{U}_1 \subset \widetilde{U}$ such that $q \in \widetilde{U}_1 \cap M$ implies that $\alpha^{-1}(q) \in V$. This gives us the desired Equation 1.

Let's see how the proof ends. Let $\widetilde{U}_1 = F(\widetilde{V}_1)$. Replace V by V_1 in Equation 1. Let G_1 be the restriction of G to $\widetilde{V}_1$ and let F_1 be the restriction of F_1 to $\widetilde{U}_1$. The maps F_1 and G_1 are diffeomorphisms and inverses of each other. Let $q \in \widetilde{U}_1 \cap M$ be any point. We know that $q = \alpha(u)$ for some $u \in V_1$. But then $q = G(u_1, ..., u_k, 0, ..., 0)$ where $(u_1, ..., u_k)$ are the coordinates of u. But then $F(q) = (u_1, ..., u_k, 0, ..., 0)$. So on $\widetilde{U}_1 \cap M$, the map α^{-1} coincides with the first k coordinates of F.