Manifolds

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The purpose of these notes is to define what is meant by a *manifold*, and then to give some examples.

1 Topological Spaces

If you haven't seen topological spaces yet, just skip this section.

The space underlying a manifold is traditionally taken to be a second-countable Hausdorff topological space. To say that a space X is *second countable* is to say that there is a countable collection of open subsets of X such that every open subset of X is a union of members from the countable collection – i.e., X has a *countable basis*. To say that X is *Hausdorff* is to say that, for every two distinct points $x, y \in X$, there are disjoint open sets U_x and U_y such that $x \in U_x$ and $y \in U_y$.

That is all I'm going to say about topological spaces. Below I'm going to define manifolds in terms of metric spaces. The definition I give is equivalent to the definition that is given in terms of topological spaces, even though at first glance it looks different.

2 Metric Spaces

A *metric space* is a set X together with a function $d : X \times X \to \boldsymbol{R}$ such that

- $d(x, y) \geq 0$ for all $x, y \in X$, with equality if and only if $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

d is called the *distance function* on X.

Example 0: It almost goes without saying, but I'll say explicitly that any subset of a metric space is automatically a metric space, with the same metric. This fact is frequently and implicitly used.

Example 1: The classic example of a metric space is a subset $X \subset \boldsymbol{R}^n$ equipped with the distance function given by $d(x, y) = \|x - y\|$, here $\| \cdot \|$ is the Euclidean norm.

Example 2: This example is unrelated to the rest of the material in the notes, but I like it. Choose a prime p and on $\boldsymbol{Z}$ define $d_(x, y) = p^{-k}$, where k is the largest integer such that p^k divides $x - y$. This is known as the p-adic metric on $\boldsymbol{Z}$. Geometrically, $\boldsymbol{Z}$ looks like a dense subset of points in a Cantor set when it is equipped with the p-adic metric.

From now on, X denotes a metric space, and d the metric on X.

Balls: Given $x \in X$ and some $r > 0$, we define

$$B_r(x) = \{y \in X | \ d(x, y) < r\}. \tag{1}$$

The set $B_r(x)$ is known as the open ball of radius r about x.

Open Sets: A subset $U \subset X$ is *open* if, for every $x \in U$, there is some $r > 0$ such that $B_r(x) \subset U$.

Continuity: Given to metric spaces X and Y, a map $f : X \to Y$ is called *continuous* if, for all open $V \subset Y$ the inverse image $U = f^{-1}(V)$ is open in X. This definition is equivalent to the usual $\epsilon - \delta$ definition of continuity. From our definition, it is clear that the composition of continuous functions is continuous. If $f : X \to Y$ and $g : Y \to Z$ are both continuous, then so is $g \circ f : X \to Z$.

Homeomorphisms: A map $f : X \to Y$ is a *homeomorphism* if f is a bijection and both f and f^{-1} are continuous. So, in particular, a homeomorphism from X and Y induces a bijection between the open subsets of X and the open subsets of Y. To test your understanding, prove that the open ball in $\boldsymbol{R}^n$ is homeomorphic to $\boldsymbol{R}^n$ but the closed ball in $\boldsymbol{R}^n$ is not.

Compactness: A *covering* of X is a collection of open sets whose union equals X. A *subcover* of a covering is some subset of the covering which is, itself, a covering. A subset of X is *compact* if every covering of X has a subcovering with finitely many elements. It is a classic theorem that a subset of $\boldsymbol{R}^n$ is compact if and only if it is closed and bounded.

σ-Compactness X is called *σ-compact* if X is a countable union of compact subsets. For instance, any closed subset of $\boldsymbol{R}^n$ is σ-compact, but only the bounded closed subsets are compact.

3 Topological Manifolds

Coordinate Charts: Let M be a metric space. A *coordinate chart* in M is an open set $U \subset M$ and a homeomorphism

$$h : \boldsymbol{R}^k \to U. \tag{2}$$

We write this as (U, h). This coordinate chart is said to *contain* p if $p \in U$. Here k could depend on the point – e.g. when M is the union of a line and a plane – but we're going to be interested in the case when k is the same for all points.

Basic Definition: A *topological k-manifold* is a σ-compact metric space M such that every point of M is contained in some coordinate chart.

Examples: Here are some examples of topological manifolds.

- $\boldsymbol{R}^n$ itself.
- S^n, the n-dimensional sphere.
- The surface of any polyhedron.
- The Koch snowflake.
- The square torus - i.e. the square with sides identified.

The simplest example of a σ-compact metric space which is not a topological manifold is the union of the coordinate axes in $\boldsymbol{R}^2$.

Overlap Functions: Suppose that M is a topological manifold. Suppose that (U_1, h_1) and (U_2, h_2) are two coordinate charts in M. Suppose that these charts overlap. That is, the set $V = U_1 \cap U_2$ is nonempty. Then we have a map

$$h_2^{-1} \circ h_1 : h_1^{-1}(V) \to h_2^{-1}(V). \tag{3}$$

This map is a homeomorphism because it is the composition of homeomorphism. The function $h_2^{-1} \circ h_1$ is called an *overlap* function.

4 Smooth Manifolds

Compatible Charts: Let M be a topological manifold. Two coordinate charts $U_1, U_2 \in M$ are *smoothly compatible* if the overlap function defined by these charts is not just a homeomorphism, but actually smooth.

Atlases: A *smooth atlas* $\mathcal{A}$ on M is a system of coordinate charts which are all compatible with each other. We insist that every point of M is contained in at least one chart of $\mathcal{A}$. The atlas $\mathcal{A}$ is called *maximal* if there is no additional coordinate chart, not in $\mathcal{A}$, which is compatible with all the coordinate charts in $\mathcal{A}$. Zorn's Lemma guarantees that every smooth atlas on M is contained in a maximal smooth atlas.

Main Definition: A *smooth manifold* is a topological manifold equipped with a maximal smooth atlas.

Example 1: Let $F : \boldsymbol{R}^n \to \boldsymbol{R}^m$ be a smooth map and let $q \in \boldsymbol{R}^m$ be some point. We call q a *regular value*, if for every $p \in F^{-1}(q)$, the differential $dF(p)$ is surjective. In this situation, the Implicit Function Theorem gives a coordinate chart about p, and this coordinate chart is smooth in the usual sense. So, when q is a regular value, $F^{-1}(q)$ is a smooth manifold of dimension $n - m$ assuming that it is nonempty.

Example 2: Take the unit cube in $\boldsymbol{R}^n$ and identify opposite sides in the most direct possible way. Call the resulting space X. If you want to make X into a metric space, define $d(x, y)$ to be the length of the shortest path joining x to y, where these paths are allowed to go through the identified sides. You can find coordinate charts from X into $\boldsymbol{R}^n$ which are *local isometries* i.e. distance preserving when restricted to small enough open sets. (Try this for $n = 2$ first.) The overlap functions are again local isometries and hence smooth. So, the unit cube in $\boldsymbol{R}^n$ with its sides identified is naturally a smooth n-manifold. It is known as the *square n-torus*.

5 Maps between Smooth Manifolds

Main Definition: Suppose that M_1 and M_2 are smooth manifolds. A map $f : M_1 \to M_2$ is *smooth* if all compositions of the form

$$h_2^{-1} \circ f \circ h_1 \tag{4}$$

are smooth, where h_1 is a homeomorphism associated to a chart in M_1 and h_2 is a homeomorphism associated to a chart in M_2. What makes this a good definition is that all the overlap functions are smooth. So, to verify the smoothness of f, you don't have to examine all the uncountably many coordinate charts in the two maximal atlases. You just to verify it for some pair of sub-atlases.

Diffeomorphisms: A map $f : M_1 \to M_2$ is a *diffeomorphism* if f is a bijection and both f and f^{-1} are smooth. It is easy to verify that the composition of smooth diffeomorphisms is again a diffeomorphism. In particular, the set of diffeomorphisms from M to itself is a group! It is written $\text{Diff}(M)$.

Exercise: Here is an interesting but somewhat difficult problem. Suppose that M is any smooth manifold and $p_1, ..., p_n \in M$ are some finite set of points. Let π be some permutation of these points. Prove that there is a diffeomorphism of M which agrees with π on these points. Try it first for $\boldsymbol{R}^2$, and then for homeomorphisms of topological manifolds. Getting the map to be smooth, on a smooth manifold, is additional work.

6 Riemann Surfaces

The same basic framework allows you to define other kinds of structures on topological manifolds. I'll just give one example, because it is especially important.

Complex Analytic Maps: Let $U \subset \boldsymbol{C}$ be an open set. A map $f : U \to \boldsymbol{C}$ is called *complex analytic* if it is continuously differentiable, and

$$df(p) = \begin{bmatrix} A(p) & B(p) \\ -B(p) & A(p) \end{bmatrix} \tag{5}$$

for all $p \in U$. The real valued functions $A(p)$ and $B(p)$ vary continuously with p. Geometrically, $df(p)$ is a similarity. When Equation 5 is written out in terms of the matrix of partial derivatives, it is known as the *Cauchy-Riemann equations*.

Alternate Formulation: It is an amazing fact that a complex analytic map is always smooth, and equal to a convergent power series

$$f(z) = \sum_{i=0}^{\infty} c_j (z - z_0)^j, \qquad c_j \in \boldsymbol{C} \tag{6}$$

in a neighborhood of each point $z_0 \in U$. You could take this as an alternate definition of what it means for a map to be complex analytic.

Main Definition: A *Riemann Surface* is a 2-dimensional smooth manifold such that all the overlap functions defined by its atlas are complex analytic.