The Cauchy-Binet Theorem

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The Cauchy-Binet theorem is one of the steps in the proof of the Matrix Tree Theorem. Here I’ll give a proof.

Let $A$ be an $n \times N$ matrix and let $B$ be an $N \times n$ matrix. Here $n < N$. The matrix $AB$ is an $n \times n$ matrix. Given any subset $S \subset \{1, ..., N\}$ having $n$-elements, form the two $n \times n$ matrices $A_S$ and $B_S$, obtained by just using the rows (or columns) indexed by the set $S$. Define

$$f(A, B) = \det(AB), \quad g(A, B) = \sum_S \det(A_S) \det(B_S).$$

The sum ranges over all choices of $S$. The Cauchy-Binet theorem is that $f(A, B) = g(A, B)$ for all choices of matrices.

Think of $A$ and $B$ each as $n$-tuples of vectors in $\mathbb{R}^N$. We get these vectors by listing out the rows of $A$ and the columns of $B$. So, we can write

$$f(A, B) = f(A_1, ..., A_n, B_1, ..., B_n),$$

and likewise for $g$.

The idea of the proof is to check that the values of $f$ and $g$ change in the same way when the list $A_1, ..., A_n$ and the list $B_1, ..., B_n$ are changed just one vector at a time. All the properties we list come from well-known properties of the dot product and the determinant.

- If $A_i$ is replaced by $\lambda A_i$ then $f(A, B)$ is replaced by $\lambda f(A, B)$.
- If $B_i$ is replaced by $\lambda B_i$ then $f(A, B)$ is replaced by $\lambda f(A, B)$.
- If $A_i$ is replaced by $\lambda A_i$ then $g(A, B)$ is replaced by $\lambda g(A, B)$.
- If $B_i$ is replaced by $\lambda B_i$ then $g(A, B)$ is replaced by $\lambda g(A, B)$. 

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Now consider the analogous operation of addition. Let $A'$ denote the list obtained from $A$ by changing the vector $A_i$ to $A'_i$. Likewise define $A''$ and $B'$ and $B''$. We only change things in the one position. Suppose $A_i = A'_i + A''_i$ and $B_i = B'_i + B''_i$. Then

- $f(A, B) = f(A', B) + f(A'', B)$.
- $f(A, B) = f(A, B') + f(A, B'')$.
- $g(A, B) = g(A', B) + g(A'', B)$.
- $g(A, B) = g(A, B') + g(A, B'')$.

In view of the fact that $f$ and $g$ transform exactly the same way under all the operations above, it suffices to consider the case when all the vectors are amongst the standard basis vectors. If $A_i = A_j$ for some pair of indices, then $\det(A_S) = 0$ for all $S$ and also $\det(AB) = 0$ because $AB$ has a repeated row. The same goes if $B_i = B_j$ for some pair of indices. So, we can assume that no two vectors of $A$ are the same and no two vectors of $B$ are the same. Call the associated matrices special.

In short, it suffices to prove the Cauchy-Binet theorem when $A$ and $B$ are special matrices. So, $A$ and $B$ are both matrices with $n$ ones and the rest zeros. The rows of $A$ are linearly independent and the columns of $B$ are linearly independent. In this situation, there are unique sets $S_A$ and $S_B$ of $n$ elements such that $\det(A_{S_A}) = 1$ and $\det(B_{S_B}) = 1$. For all other sets we get zero. So $g(A, B) = 1$ if $S_A = S_B$ and otherwise $g(A, B) = 0$. When $S_A = S_B$, the matrix $AB$ is the identity so $f(A, B) = 1$. Otherwise, $AB$ has at most $n-1$ nonzero entries. Hence $f(A, B) = 0$. So, in all cases $f(A, B) = g(A, B)$.

This completes the proof.