Congruence Subgroups and Platonic Solids

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1 Mobius Transformations

A *Mobius transformation* is a map of the form

$$T(z)=\frac{az+b}{cz+d}, \qquad \det\begin{bmatrix} a & b \\ c & d \end{bmatrix}\neq 0, \qquad a,b,c,d\in \boldsymbol{C}. \tag{1}$$

Below we'll be more restrictive about these requirements. Here z is a complex number, but it also makes sense to take $z=\infty$. In this case, $T(\infty)=a/c$. If $c=0$ then $T(\infty)=\infty$. It also makes sense to say that $T(z)=\infty$. This happens if $z=-d/c$.

Here are some pretty easy facts about Mobius transformations.

- The Mobius transformations form a group.
- If $a,b,c,d\in \boldsymbol{R}$ and if $ad-bc=1$ then T maps points on $\boldsymbol{R}\cup\infty$ to points on $\boldsymbol{R}\cup\infty$. For instance $T(z)=-1/z$ maps 0 to ∞ and ∞ to 0. With the same conditions, T maps the upper half plane in $\boldsymbol{C}$ to itself.
- A Mobius transformation preserves angles between curves. That is, if A and B are curves that meet at some angle, then $T(A)$ and $T(B)$ meet at the same angle.
- A Mobius transformation maps a generalized circle to a generalized circle. A generalized circle is either a circle or a straight line (union ∞).
- A Mobius transformation is determined by what it does to and 3 distinct points. So, if $S(p_j)=T(p_j)$ for $j=1,2,3$ then $S=T$.

2 The Modular Group

$SL_2(\boldsymbol{Z})$ is the group of 2×2 integer matrices with determinant 1. These elements have the form

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad a, b, c, d \in \boldsymbol{Z}, \qquad ad - bc = 1. \tag{2}$$

This is arguably the most famous group in mathematics.

There is a variant which is even nicer. The group $PSL_2(\boldsymbol{Z})$ is the group obtained from $SL_2(\boldsymbol{Z})$ by declaring that the two matrices M and $-M$ are equivalent. The set of these equivalence classes is still a group because the 4 products

$$(A)(B), \qquad (-A)(B), \qquad (A)(-B), \qquad (-A)(-B)$$

are all equivalent.

$PSL_2(\boldsymbol{Z})$ has two advantages over $SL_2(\boldsymbol{Z})$. First, the two matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \tag{3}$$

have orders 4 and 6 in $SL_2(\boldsymbol{Z})$ but have orders 2 and 3 in $PSL_2(\boldsymbol{Z})$. What is going on is that

$$A^2 = B^3 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4}$$

We'll see that $[A]$ and $[B]$ generate $PSL_2(\boldsymbol{Z})$.

A second reason that $PSL_2(\boldsymbol{Z})$ is nicer has to do with Mobius transformations. The element given in Equation 2 gives rise to a Mobius transformation T_M whose formula is

$$T_M(z) = \frac{az + b}{cz + d}. \tag{5}$$

Note that $T_M = T_{(-M)}$. So, in terms of Mobius transformations, there is a redundancy in $SL_2(\boldsymbol{Z})$ that we treat by considering $PSL_2(\boldsymbol{Z})$ as well. Each element of $PSL_2(\boldsymbol{Z})$ gives rise to a different Mobius transformation.

The assignment of Mobius transformations to elements of $PSL_2(\boldsymbol{Z})$ has a nice property. A calculation shows that

$$T_{AB} = T_A \circ T_B. \tag{6}$$

If you know about group homomorphisms, we can say that the map $M \to T_M$ is an injective homomorphism from $PSL_2(\boldsymbol{Z})$ into the group of all Mobius transformations.

Lemma 2.1 *$PSL_2(\boldsymbol{Z})$ is an automorphism of the Farey graph.*

Proof: Given that Mobius transformations preserve angles and map generalized circles to generalized circles, it suffices to prove that each element of $PSL_2(\boldsymbol{Z})$ maps vertices of the Farey graph to vertices of the Farey graph and preserves the edge relation. The first fact comes from the fact that the entries of the elements in the group are integers. The second fact comes from the product formula for the determinant. This is a tedious but not hard calculation. ♠

Lemma 2.2 *The elements A and B generate $PSL_2(\boldsymbol{Z})$.*

Consider the Farey triangulation of the upper halfplane. This is the triangulation from the first HW assignment. You join each integer point to ∞ using a vertical ray and you join the rationals p_1/q_1 and p_2/q_2 by a semi-circular edge iff

$$\det \begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix} = \pm 1.$$

The group $PSL_2(\boldsymbol{Z})$, acting on the upper half plane as Mobius transformations, preserves the Farey graph. The element A has the effect of swapping the triangles $(0, 1, \infty)$ and $(0, -1\infty)$. In terms of the dual tree, this element reverses an edge of the tree and then the rest of the tree goes along for the ride. The element B fixes a point in the $(0, 1, \infty)$ triangle and just rotates this triangle. In terms of the dual tree, B fixes a vertex and rotates the branches of the tree around this vertex.

Given any triangle T in the Farey graph, you can find a path of triangles connecting T to the $(0, 1, \infty)$ triangle T_0 using a finite chain of adjacent triangles. This is just a path in the dual tree. By induction on the length of the finite chain, you can find some work in A and B which maps T to T_0. But then we can compose with a suitable power of B to map T to T_0 in any of the 3 possible orientation-preserving ways. Given $g \in PSL_2(\boldsymbol{Z})$ let $T = g(T_0)$. We can find some product w of the generators such that $w(T) = T_0$. But then gw fixes the vertices of T_0 and must be the identity. Hence $g = w^{-1}$. This expresses an arbitrary element of $PSL_2(\boldsymbol{Z})$ as a product of generators. ♠

3 Congruence Subgroups

Given an integer N, we define $\Gamma_N \subset PSL_2(\boldsymbol{Z})$ to be those equivalence classes of matrices M such that $M \equiv \pm I \bmod N$. Here I is the identity matrix. For example, Γ_2 consists of those (equivalence classes of) matrices such that the diagonal entries are odd and the off-diagonal entries are even.

Lemma 3.1 Γ_N *is a group.*

Proof: Let A_N denote the matrix obtained by reducing the entries of A mod N. The elements of A_N are elements of $\boldsymbol{Z}/N$. Likewise define B_N. Because one can do both addition and multiplication in $\boldsymbol{Z}/N$ (i.e., because it is a *ring*) we have

$$(AB)_N = A_N B_N$$

When $A, B \in \Gamma$ we have $A_N = \pm I$ and $B_N = \pm I$, but then $(AB)_N = \pm I$.

There is a similar proof that works for inverses, but actually for inverses we can see it more directly. There is a nice formula for the inverse of an element of $SL_2(\boldsymbol{Z})$:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \Rightarrow \qquad A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{7}$$

You can check the formula just by multiplying things out. This formula immediately implies that $A \in \Gamma_N$ if and only if $A^{-1} \in \Gamma_N$. ♠

Now we're going to look harder at the Farey triangulation. We declare two Farey triangles T_1 and T_2 *equivalent* exactly when there is some $g \in \Gamma_N$ so that $g(T_1) = T_2$. The fact that Γ_N is a group makes this an equivalence relation: If $g(T_1) = T_2$ then $g^{-1}(T_2) = T_1$. If $g(T_1) = T_2$ and $h(T_2) = T_2$ then $hg(T_1) = T_3$.

Lemma 3.2 *Adjacent triangles are never equivalent.*

Proof: Suppose that $g \in \Gamma_N$ and $g(U_1) = U_2$ where U_1 and U_2 share an edge. Let T_1 and T_2 respectively be the $(0, 1, \infty)$ and $(0, -1, \infty)$ triangle. We can find some element h of $PSL_2(\boldsymbol{Z})$ such that $h(T_j) = U_j$ for $j = 1, 2$. But then $h^{-1}hg$ and the generator A do the same thing to T_0 and T_1. This implies that

$$h^{-1}gh = A.$$

Since $A \notin \Gamma_N$, we get that $h^{-1}gh \notin \Gamma_N$. On the other hand, reducing mod N, we have

$$(h^{-1}gh)_N = (h^{-1})_N(\pm I)h_N = \pm(hh^{-1})_N = \pm I.$$

This shows that $h^{-1}gh \in \Gamma_N$ after all. This gives a contradiction. ♠

Remarks:
(i) For those of you who know, the last calculation shows that Γ_N is a normal subgroup of $PSL_2(\boldsymbol{Z})$.
(ii) A similar argument shows that there is no element of Γ_N which maps a Farey triangle to itself in a nontrivial way.

4 Building Surfaces and Triangulations

We color the Farey triangles according to their equivalence classes. Triangles in the same class get the same color and triangle in different classes get different colors. For instance, when $N = 2$, there are only two equivalence classes. This gives the 2-coloring in which the triangles alternate colors.

We can build a surface Σ_N out of the colored Farey graph. Choose one triangle of each color and glue your choices together according to how the triangles are glued together in the upper half plane. To be precise about this, suppose that you have two pairs (T_1, e_1) and (T_2, e_2). We think of the edges as oriented. Should you glue them (according to the orientations)?

We look for an element $M \in \Gamma_N$ so that $M(e_2) = e_1$ (as oriented edges) and $M(T_2)$ shares the edge e_1 with T_1. If this works, we make the gluing. This procedure is independent of all choices because Γ_N acts as a group of symmetries of the colored Farey graph, and no element of Γ_N permutes the edges of a single triangle.

Example: Let's work out Σ_2. We know that Σ_2 is made from 2 triangles, namely $T_1 = (0, -1, \infty)$ and $T_2 = (0, 1, \infty)$. (We'll name triangles by their vertices.) The edge $(0, \infty)$ is already common to both triangles.

Let's prove that we should glue $(T_1, (0, -1))$ to $(T_2, (0, 1))$. Consider the element

$$M = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

This matrix belongs to Γ_2 and the corresponding Mobius transformation is

$$M(z) = \frac{z}{-2z+1}. \tag{8}$$

We check that $M(0) = 0$ and $M(1) = -1$. Hence $M(e_2) = e_1$, with the orientations given by the ordering of the vertices. Also, $M(\infty) = -1/2$. Hence $T_2' = M(T_2)$ is the triangle $(0, -1/2, -1)$, and indeed T_2' and T_1 meet across e_1. So, yes, we should make the gluing.

Finally, a similar argument shows that we should glue $(T_1, (-1, \infty))$ to $(T_2, (1, \infty))$. Figure 1 shows the general situation. The surface you get is made from gluing two triangles on top of each other. It is a sphere! (Some people like to delete the 3 vertices and then speak of a 3-punctured sphere.)

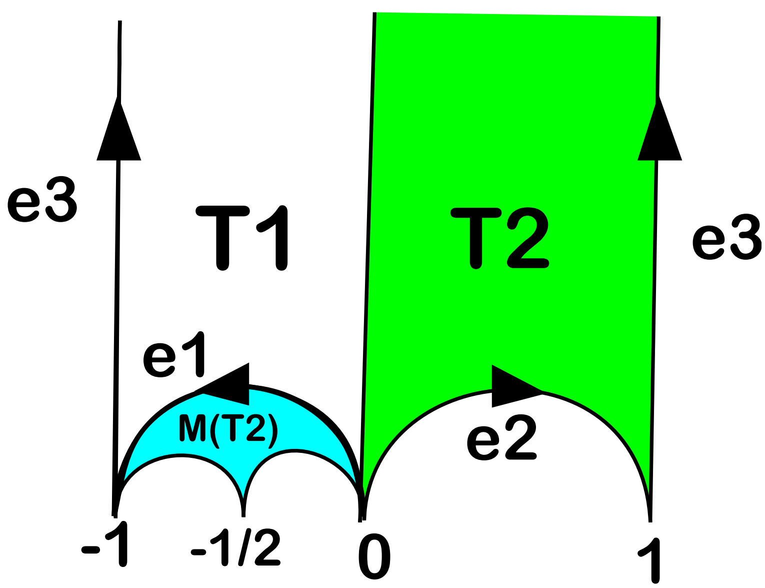

Figure 1: The surface obtained from Γ_2.

It turns out that the other small values of N yield familiar examples:

- Σ_3 is a sphere obtained by gluing together 4 triangles. The triangulation is that of the regular tetrahedron.
- Σ_4 is a sphere obtained by gluing together 8 triangles. The triangulation is that of the regular octahedron.
- Σ_5 is a sphere obtained by gluing together 20 triangles. The triangulation is that of the regular icosahedron.

I'll explain why this is what you get in the next section.

5 Symmetry of the Construction

Let Σ_N be the surface obtained by the gluing construction above. Let γ be any element of $PSL_2(\boldsymbol{Z})$. The element γ permutes the triangles of the Farey graph but might change the colors if $\gamma \notin \Gamma_N$. The crucial lemma is that Γ respects the instructions used to build the surface.

Lemma 5.1 *Let (T_1, e_1) and (T_2, e_2) be two triangle-edge pairs which are supposed to be glued together. Then are also supposed to be glued together.*

Proof: There is an element $M \in \Gamma_N$ such that $M(T_2)$ is adjacent to T_1 and has the common edge $e_1 = M(e_2)$ in common. But then $\gamma M \gamma^{-1}$ does the same thing for the pairs $(\gamma(T_1), \gamma(e_1))$ and $(\gamma(T_2), \gamma(e_2))$. We just have to show that $\gamma M \gamma^{-1} \in \Gamma_N$. We compute

$$(\gamma M \gamma^{-1})_N = \gamma_N M_N (\gamma^{-1})_N = \gamma_N (\pm I) \gamma_N^{-1} = \pm(\gamma\gamma^{-1})_N = \pm I.$$

This does it. ♠

Since one can map any element of the Farey graph to any other element, we see that there is a symmetry of the surface Σ_N which maps one triangle on Σ_N to any other triangle. Also, the elements A and B give symmetries of Σ_N. Combining this with the symmetry we already have, we see that there is a symmetry of Σ_N which rotates any given triangle by $2\pi/3$ degrees, and there is another symmetry which rotates around any given edge. This is one explanation of the great symmetry seen in the platonic solids.

Lemma 5.2 *The triangulation of Σ_N is regular of degree N.*

Proof: Thanks to the symmetry group, the triangulation has the same degree at every vertex. You can map each vertex to each other using a symmetry so they all have the same degree. Call a triangle *tall* if it has ∞ as a vertex. The element

$$\begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix}$$

belongs to Γ_N and shifts the tall triangles by N. Moreover, the tall triangles which are less than N away are inequivalant. So, you can always use N tall triangles in a row as part of your material for Σ_N. The rules above tell us to glue the first and last sides. This tells us that the degree of the triangulation corresponding to the vertex ∞ is N. Hence, every vertex has degree N. ♠

In the case $N = 3, 4, 5$ there is only one regular triangulation of the sphere having degree N, and this is why we get the Platonic solids.

What is interesting here is that the surface Σ_N retains this huge amount of symmetry when $N > 5$. The surface Γ_7 turns out to be a triangulation of a 3-holed surface made from 56 triangles. The triangulation has 24 vertices and 84 edges. Its group of symmetries coming from $PSL_2(\boldsymbol{R})$ has order 168.