Math 123 HW 1

1. Use the Matrix Tree Theorem to prove that the complete bipartite graph $K_{n,n}$ has n^{2n-2} spanning trees.

2. (This problem and the next are essentially Problem 2.2.13 in the book.) Let T be a tree whose vertices are colored black and white so that same-colored vertices are never adjacent. Suppose that T has n black vertices and also n white vertices. Prove that T has at least one white leaf and one black leaf.

3. Suppose you have a tree of the kind in Problem 2. You can label the black vertices $1, ..., n$ and the white vertices $1, ..., n$. You can then make a code, as follows. Delete the edge and vertex of the least black leaf and let w_1 be the label of the white vertex where it attaches. Do the same for the the least white leaf and let b_1 be the label of the black edge where it attaches. This gives you a pair (w_1, b_1).

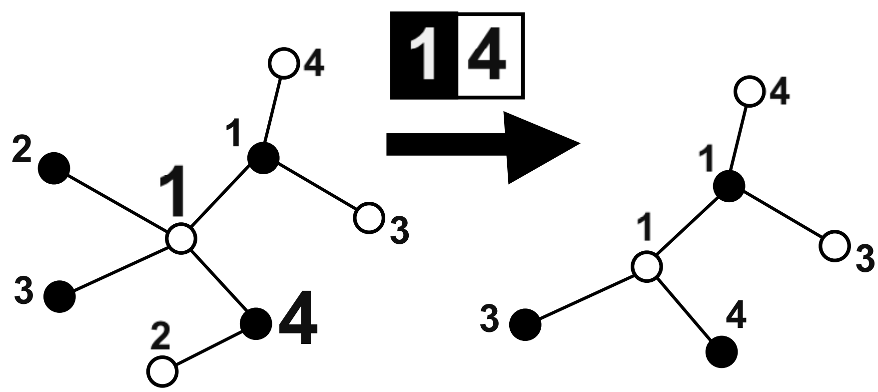

Figure 1: A variant of the Prufer code

In the example in Figure 1 the pair is $(1, 4)$. Continue in this way until you are left with just a single edge. (Basically, you are doing the Prufer construction, but you are doing it with both hands.) This gives you a string $w_1, b_1, w_2, b_2, ..., w_{n-1}, b_{n-1}$. Prove that this operation gives a bijection between the set of such trees and the set of sequences in $\{1, ..., n\}$ having length $2n-2$. This gives another proof that $K_{n,n}$ has n^{2n-2} spanning trees.

4. The relevance of this problem will emerge when you get to the next one. Consider the Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, \ldots$ These numbers satisfy the "initial conditions" $a_1 = a_2 = 1$ and the "difference equation"

$$a_{n+2} = a_{n+1} + a_n. \tag{1}$$

The two solutions of the equation $x^2 = x + 1$ are

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

Call these numbers λ_1 and λ_2. Prove that there are constants C_1 and C_2 such that

$$a_n = C_1 \lambda_1^n + C_2 \lambda_2^n,$$

for all n. Hint: Show that any linear combination like this also satisfies Equation 1 and then adjust the constants to get the initial conditions right. Since $|\lambda_2| < 1$, you get that $a_n \sim C_1 \lambda_1^n$ as $n \to \infty$. Here the symbol $\sim$ means "within one of each other" in this case.

If you've had an O.D.E. class, you might appreciate the analogy between the setup here and the uniqueness result for second order linear O.D.E.s

5. (This is essentially Problem 2.2.15 from the book.) Let $G_1, G_2, G_3, G_4 \ldots$ be the sequence of graphs shown in Figure 2.

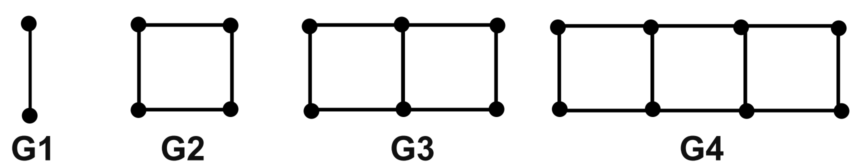

Figure 2: A sequence of graphs

Let b_n denote the number of spanning trees of G_n. Prove that

$$b_{n+2} = 4b_{n+1} - b_n. \tag{2}$$

Then deduce that there is some constant λ such that $b_n \sim C\lambda^n$ for n large. What is the value of C?

6. Let $C(k, n)$ denote the number of labeled trees with n vertices which have all degrees equal to either k or 1. As $n \to \infty$, which is larger, $C(3, n)$ or $C(4, n)$?