Math 123 HW 4

1. This is problem 6,1.16 in the book. Prove that any planar Eulerian graph can be drawn in such a way that the pencil never crosses what has already been drawn and never retraces an edge. Figure 1 shows an example. The drawing in red has been lifted off the graph a bit so as to reveal how it goes.

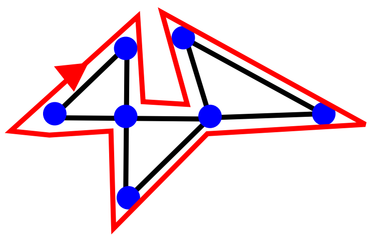

Figure 1: A planar Eulerian graph

2. Prove that the Peterson graph is not planar but that it can be drawn in the projective plane without edge crossings. The first part is a simplification of Problem 6.2.2 in the book. (See also problem 6.1.30.) The left half of Figure 2 shows the Peterson graph. The right half shows one way to view the projective plane.

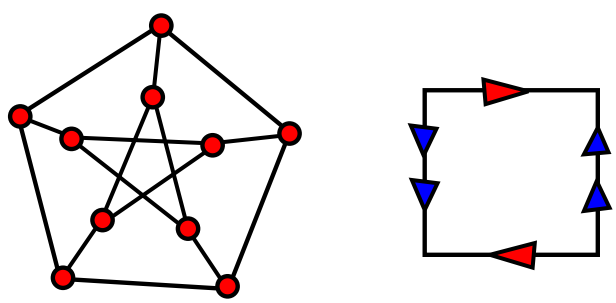

Figure 2: The Peterson graph and the projective plane.

There are various ways to think of the projective plane. If you want to take a direct approach to this problem, then you could think of the projective plane as a square with its opposite sides identified "crossways" as shown in the picture. If you want to think about a really beautiful solution to this problem – and it is worthwhile to try for it – think about the projective plane as the sphere with antipodal (i.e. opposite) points identified and consider the dodecahedron graph.

3. Consider the graph whose vertices are the vertices of the n dimensional cube and whose edges are the edges of the n dimensional cube. Prove that the genus of this graph tends to ∞ as n tends to ∞. In other words, there is no single surface on which you can draw these graphs.

4. Do problem 6.1.33 in the book. That is, suppose that G is a triangulation, and let n_i be the number of vertices of degree i in G. Prove that $\sum(6-i)n_i = 12$.

Remark: This is not part of the problem, but I can't resist saying something about it. This formula has a great geometric interpretation in the special case that $n_i = 0$ for all $i = 7, 8, 9, ...$ In this case, you can build G out of equilateral triangles and the result will be isometric to the boundary of a convex polyhedron. The quantity $6 - \deg(v)$ measures the difference between 2π and the "cone angle" at the vertex v. One can view this number as a kind of curvature, concentrated at the vertices and then the formula says that the total curvature is $4\pi = 12 \times \pi/3$. This result is in turn a special of the Gauss-Bonnet formula from differential geometry. So, in a sense, this problem is giving a combinatorial version of the Gauss-Bonnet formula.

5. Prove that every triangulation has an embedding in which the edges are straight line segments. Hint: Consider the counterexample with the fewest edges and then look at the following two cases:

1. There is an interior vertex – i.e. not on the outer cycle – which has degree 3.

2. All interior vertices have degree at least 4.

Then study what happens when you selectively delete or contract edges. Each of these cases breaks down into a few subcases.