Math 123 HW 9

1. Prove that the line and the plane are not quasi-isometric to each other.

2. In class I mentioned how the question of showing that one could move any Farey triangle to another one using certain elements of $PSL_2(\boldsymbol{Z})$ was really just a question about trees. So, here is the tree question. Let T be the infinite trivalent tree, drawn in the plane. (If you like, think of T as the dual to the Farey graph.) Let A be the automorphism of T which geometrically rotates around the center of an edge e of T. Let B be the the automorphism of T which geometrically rotates one third of the way around around one of the vertices of T that is incident to T. So, A^2 and B^3 are the identity. Prove, for any two vertices $v, w \in T$, there is some composition of A and B (e.g. $ABABBA$) which maps v to w.

3. Consider the following union H of 4 triangles in the Farey graph:

$$(-1, 0, \infty), \qquad (0, 1, \infty), \qquad (1, 2, \infty), \qquad (0, 1/2, 1).$$

Figure 1 shows a schematic picture of H. Find matrices of Γ_3 – i.e. 2×2 integer matrices congruent to $\pm I$ mod 3 which identify the edges of H in the pattern shown in Figure 2. This is part of the proof that the surface Σ_3 is a sphere triangulated into the pattern of a regular tetrahedron. (Hint: you just have get the elements to act the right way on the vertices.)

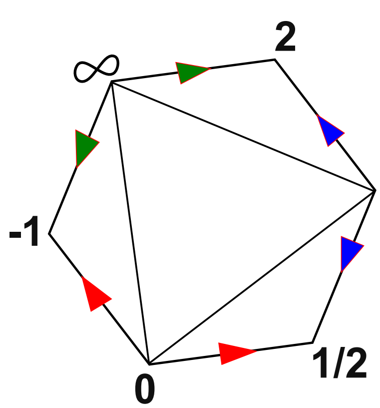

Figure 1: The hexagon H.

4. Figure 2 shows a picture of (part of) the infinite affine graph. (All edges are supposed to have length 1.) Prove that there are constants C_1 and C_2 such that any ball of radius N in the graph has at least $C_1^N - C_2$ disjoint balls of radius 100 inside it. The same result holds when 100 is replaced by any K, but the constants C_1 and C_2 would change. This fact is the first step in the proof that the affine graph is not quasi-isometric to the Euclidean plane. For what it is worth, the affine graph is quasi-isometric to the hyperbolic plane.

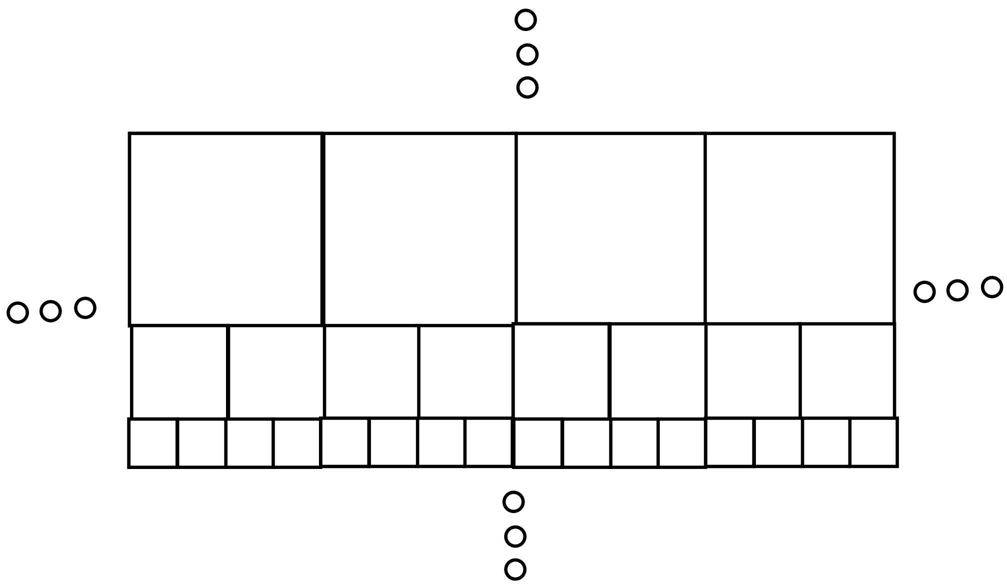

Figure 2: The affine graph

5. Say that a *grid walk* is a walk in the plane which only uses the edges of the square grid. All grid walks must start at $(0,0)$ but they can end at other points. Call two grid walks equivalent if they have the same endpoints and the loop formed by the two walks has signed area 0. For instance, Figure 3 shows two equivalent grid walks.

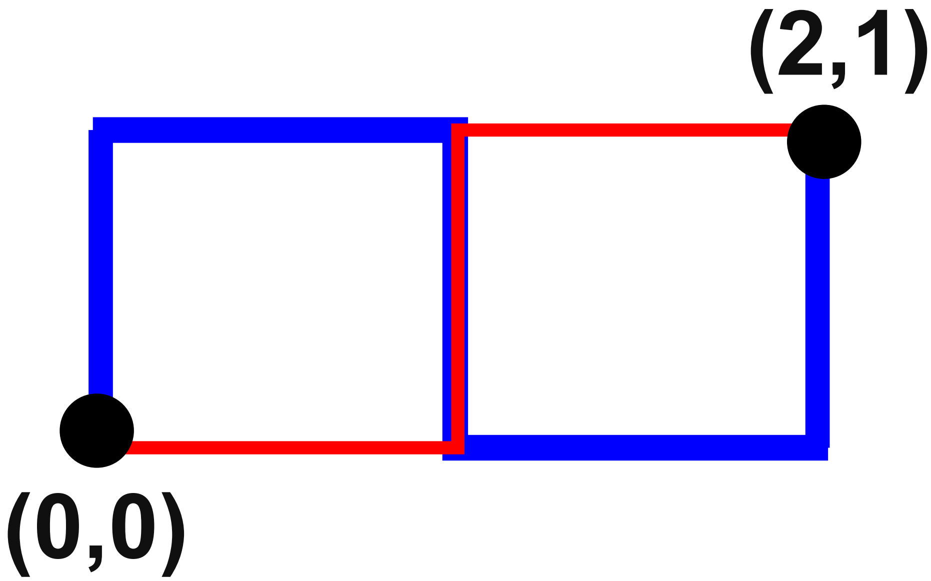

Figure 3: Equivalent grid walks

Form a graph whose vertices are equivalence classes of grid walks and whose edges are as follows. Two equivalence classes are joined by an edge if they have representatives that differ by a single edge. Also, two equivalence classes are joined by an edge if they have the same endpoints and the signed area of the loop that they make has area ± 1. So, every vertex of this graph has degree 6. Prove that this graph is connected. This graph is a copy of the Heisenberg graph.