Some Infinite Graphs

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In these notes, I'll describe 3 nice infinite graphs. The first one is not the Cayley graph of a group but it is highly symmetric just the same. The second two are Cayley graphs of groups, though I will somewhat downplay the group theory.

1 The Coarse Hyperbolic Plane

This infinite graph has the following description. Start with a vertical ray divided into segments of lengths $...4, 2, 1, 1/2, 1/4, ...$ Attach an infinite horizontal row of squares having side length 2^s to the edge of the vertical ray which has length 2^s. Figure 1 shows part of the picture. The picture is meant to extend in all directions. The union of squares fills the upper half plane.

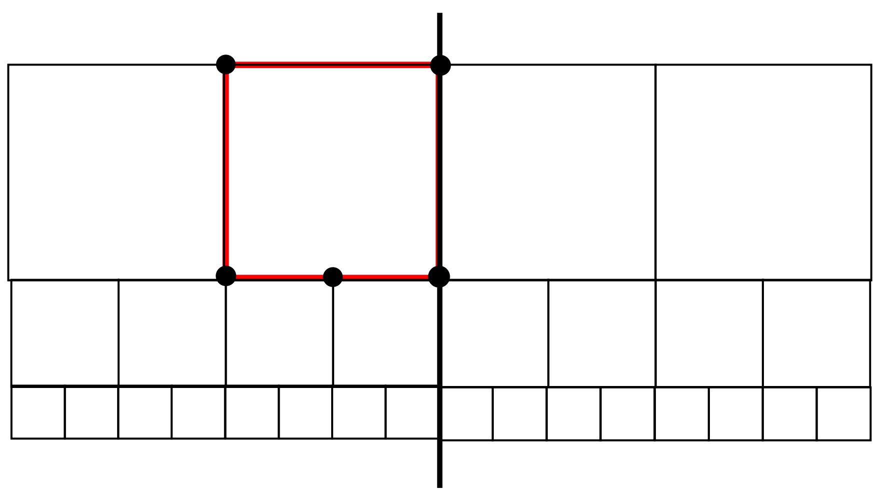

Figure 1: The coarse hyperbolic plane

This graph is an infinite union of 5-cycles. One of the 5 cycles is highlighted in the figure. This graph is sometimes called the *coarse hyperbolic plane*. We put a metric on the coarse hyperbolic plane by declaring that all its edges have length 1. The distance between any two points is defined to be the length of the shortest path which joins them. Let's compare the coarse hyperbolic plane to the hyperbolic plane.

The *hyperbolic plane* is the upper half plane equipped with a funny way of measuring distances. At the point (x, y) the inner product of two vectors V and W is defined by the formula

$$\langle V, W\rangle_{(x,y)} = \frac{1}{y^2}(V \cdot W). \tag{1}$$

In particular the langth of the vector V at (x, y) is

$$\|V\|_{(x,y)} = \frac{1}{y}\sqrt{V \cdot V}. \tag{2}$$

This way of measuring lengths of vectors is called the *hyperbolic metric*. The hyperbolic plane is denoted $\boldsymbol{H}^2$. The length of a parametrized curve $\alpha : [0, 1] \to \boldsymbol{H}^2$ is defined to be

$$\text{length}(\alpha) = \int_0^1 \|\alpha'(t)\|_{\alpha(t)}\ dt. \tag{3}$$

The distance between two points in the hyperbolic plane is defined to be the infimum of the lengths of curves joining these points. In general, this kind of definition might not work to define a metric space, but in this case it does. The shortest curves joining two points in $\boldsymbol{H}^2$ are either vertical line segments or else arcs of circles which meet the real axis at right angles.

A map: $f : \boldsymbol{H}^2 \to \boldsymbol{H}^2$ is an *isometry* if it preserves the distances between points. It is easy to check that $f(x, y) = (x + t, y)$ and $g(x, y) = (rx, ry)$ are isometries for all choices of t and r. If you draw the coarse hyperbolic plane inside $\boldsymbol{H}^2$ then all the 5-cycles have the same size. Using the maps just described you can find an isometry from one 5-cycle to any other. This means that the edges of the graph, when drawn in $\boldsymbol{H}^2$, are all the same length to within some factor of K. Probably you can take $K = 2$.

This is the beginning of the proof that the coarse hyperbolic plane is quasi-isometric to $\boldsymbol{H}^2$. This is why it is called the coarse hyperbolic plane. There are about C^r distinct points inside the disk of radius r in the coarse hyperbolic plane. This is the beginning of the proof that the coarse hyperbolic plane is not quasi-isometric to the Euclidean plane.

2 The Heisenberg Graph

We introduce 3 symbols A, B, C and consider the group of words in these symbols, subject to the relations that

$$AC = CA, \quad BC = CB, \quad ABA^{-1}B^{-1} = C.$$

Here A^{-1} is such that $AA^{-1} = A^{-1}A = e$, the empty word. There are other relations implied by these. For instance

$$BAB^{-1}A^{-1} = C^{-1}, \qquad ABA^{-1}B^{-1}C^{-1} = e.$$

Two finite words are declared equivalent if there is a finite number of substitions of the relations above which brings the one word into the other. For instance

$$CBA \sim CBA(A^{-1}B^{-1}ABC^{-1}) \sim CABC^{-1} \sim CC^{-1}AB \sim AB.$$

In general, any word in the Heisenberg group is equivalent to

$$A^aB^bC^c, \qquad a, b, c \in \boldsymbol{Z}.$$

The *Heisenberg graph* is the Cayley graph $\Gamma(G, S)$ where G is the Heisenberg group and S is the generating set $\{A, B, C, A^{-1}, B^{-1}, C^{-1}\}$. When we draw $\Gamma(G, S)$ we use the convention of omitting the edges labeled by A^{-1}, B^{-1}, C^{-1}. The idea is that if ew go backwards along the edge labeled X we are doing X^{-1}.

Let Γ be the Heisenberg graph. Let Γ_0 be the $\boldsymbol{Z}^2$ grid. There is a map $\pi : \Gamma \to \Gamma_0$ which just collapses the C edges. More precisely,

$$\pi(A^aB^bC^c) = (a, b).$$

Figure 2 shows the kind of corkscrew picture which maps to a single square in $\boldsymbol{Z}^2$.

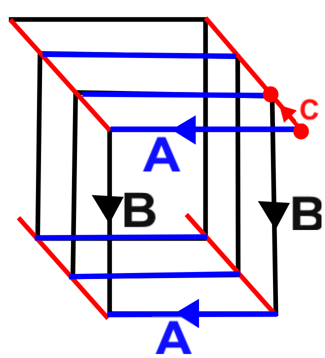

Figure 2: One corkscrew in the Heisenberg graph

The way you get the Heisenberg graph is that you build one corkecrew per unit square in the $\boldsymbol{Z}^2$ graph and glue these corkscrews together along their edges.

The Heisenberg graph has some obvious symmetries. The map which sends the vertex $A^aB^cC^c$ to $A^aB^aC^{c+1}$ respects the edge relations and gives a graph automorphism. More generally, the map $T_n : \Gamma \to \Gamma$ given by

$$T^n(A^aB^bC^c) = A^aB^bC^{c+n} \tag{4}$$

gives a graph automorphism for any $n \in \boldsymbol{Z}$. We call these automorphisms *vertical translations*.

Let γ be a walk in Γ which only uses A and B edges. We say that γ is a *lift* of the path $\gamma_0 = \pi(\gamma)$. Each path γ_0 in the Γ_0 has infinitely many lifts, but they all differ by vertical translations. Suppose γ_0 is a closed loop and γ is some lift. Then π maps the two endpoints of γ to the same point. So, these endpoints differ just by c^k. In other words, one endpoint is $A^aB^bC^c$ and the other is $A^aB^bC^{c+k}$. We call k the *vertical displacement* of the lift.

Lemma 2.1 *The vertical displacement of a lift of a closed loop equals the signed area of the loop.*

Proof: Let A be the signed area enclosed by γ_0. Let $v(\gamma_0)$ denote the vertical displacement of a lift of γ. We want to prove that $v(\gamma_0) = A$. The proof goes by induction on the absolute value of the number of squares enclosed by γ_0. We can write $\gamma_0 = \alpha_0\beta_0$, where α_0 encloses one fewer square and β_0 winds once around the square. The direction which β_0 winds around depends on the loop γ_0. Let's suppose that it winds counterclockwise, so that (by convention) $v(\beta_0) = 1$. Then $v(\alpha_0) = A - 1$ Using the fact that Γ is a group, we have $v(\gamma) = v(\alpha) + v(\beta)$, where α and β are lifts of α_0 and β_0. Hence $V(\gamma_0) = 1 + (A - 1) = A$. ♠

Using this lemma you can see that a path from $A^0B^0C^0$ of length $8n$ can reach any point of the form $A^aB^cC^c$ where $|a| < n$ and $|b| < n$ and $|c| < n^2$. Hence, the number of points of Γ in the ball of radius n is on the order of n^4. By symmetry – i.e. using the fact that the Heisenberg group acts as a group of automorphisms of Γ – we see that the same result holds for balls around any point of Γ. This is the beginning of the proof that Γ is not quasi-isometric to $\boldsymbol{R}^3$.

3 SOL

The graph called SOL is based on the choice of a 2×2 matrix in $SL_2(\boldsymbol{Z})$ which has one eigenvalue greater than 1 and one less than 1. Any two choices lead to quasi-isometric graphs. So, we'll work with the matrix

$$M = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \tag{5}$$

We form a graph whose vertices are $\boldsymbol{Z}^3$. First, we add the edges

$$(a, b, k) \leftrightarrow (a \pm 1, b, k), \qquad (a, b, k) \leftrightarrow (a, b \pm 1, k).$$

This creates an infinite stack of copies of the square grid. Next, we add the edges

$$(a, b, k) \to (M(a, b), k + 1). \tag{6}$$

For instance, there is an edge joining $(1, 1, 0)$ to $(2, 3, 1)$ because we have $M(1, 1) = (2, 3)$. One can realize SOL as the Cayley graph of a group. Rather than take this point of view, we'll just exhibit lots of symmetries of SOL.

Lemma 3.1 *One can map any vertex of SOL to any other vertex by an automorphism.*

Proof: Let's change notation so that a vertex of SOL is denoted (V, n) where $V \in \boldsymbol{Z}^2$ is a vector. Note that $(V, n) \leftrightarrow (W, n + 1)$ if and only if $(V, m) \leftrightarrow (W, m + 1)$. For this reason, the map $T(V, m) = (V, n)$ is an automorphism for any integers m and n. In particular, you can use an automorphism to move any point in SOL to a point of the form $(V, 0)$. Now, choose some vector W and define

$$T(V, n) = (V + M^n(W), n).$$

This is also an automorphism. In each copy of $\boldsymbol{Z}^2$ the map is just a translation. Also, the vector

$$T(V, n) = (V + M^n(W), n)$$

is connected to

$$(M(V) + M^{n+1}(W), n + 1) = T(M(V), n + 1)$$

and the vector (V, n) is connected to $(M(V), n+1)$. This shows that T is a graph automorphism. Setting $W = V$, we can map $(V, 0)$ to $(0, 0)$ by an automorphism. Since we can map any point to $(0, 0)$ using an automorphism, we can map any point to any other using an automorphism. ♠

The interesting thing about SOL is that the balls of radius n have about C^n points in them for some $C > 1$. I'll sketch how this is proved. By symmetry, it suffices to check this for balls centered at the origin. The proof is going to be slightly unusual in that we're going to slice SOL by a plane that does not quite contain vertices of SOL. We think of the vertices of SOL as being $\boldsymbol{Z}^3$ and the edges as being straight line segments. When drawn in space, these segments are generally very long, but in the natural metric they all have unit length.

Let E be the eigenvector of M corresponding to the eigenvalue greater than 1. Let Π be the plane spanned by the Z axis and the line through the origin parallel to E. Let $\widehat{\Pi}$ denote the slab of thickness 1 centered on Π. That is, $\widehat{\Pi}$ is the set of all points which are at most one unit from Π in the Euclidean metric. The way to picture $\widehat{\Pi}$ is that is intersect each horizontal plane $R \times \{m\}$ in an infinite strip of width 2. Any portion of this strip having length L intersects about L points of $\boldsymbol{Z}^2$.

Given that $M(L) = L$ and M expands distances by a factor of $\lambda > 1$. We see that all the points in

$$(\boldsymbol{Z}^2 \times \{m\}) \cap \widehat{\Pi} \tag{7}$$

which are within λ^m of the origin can be reached by a path of length about m. But this already gives about λ^m points in the ball of radius m about the origin.

It turns out that $\widehat{\Pi}$ intersects SOL in a graph that is quasi-isometric to the hyperbolic plane. Fattened up planes parallel to the other eigenvector also intersect SOL in graphs which are quasi-isometric to the hyperbolic plane. So, SOL has these two different directions in which you can slice it and get something quasi-isometric to the hyperbolic plane.

There is quite a bit more to say about the geometry of SOL, but these notes are just an introduction.