Math 1410: Fundamental Theorem of Algebra: The purpose of these notes is to give a proof of the Fundamental Theorem of Algebra in a way that relies mainly on compactness. For the proof, the main thing you need to know is that when you multiply complex numbers together, the norms multiply and arguments add. This fact is used repeatedly.

Let $\boldsymbol{C}$ denote the set of complex numbers. A *complex polynomial* is an expression of the form

$$P(z) = c_0 + c_1 z + ... + c_n z^n, \tag{1}$$

with the coefficients and variable in $\boldsymbol{C}$. We take $c_n \neq 0$ and call n the *degree* of P. A *root* of P is a value $a \in \boldsymbol{C}$ such that $P(a) = 0$.

Theorem 0.1 (Fundamental Theorem of Algebra) *Every complex polynomial of degree at least* 1 *has a root.*

The *norm* $|z|$ of a complex number $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}. \tag{2}$$

Geometrically, $|z|$ is the distance from z to the origin. The function $f(z) = |z|$ is continuous. The function P is a continuous map from $\boldsymbol{C}$ to $\boldsymbol{C}$. Likewise the function $|P| = f \circ P$, the composition of continuous maps, is a continuous map from $\boldsymbol{C}$ to $\boldsymbol{R}$.

Lemma 0.2 *If P has no root, then $|P|$ achieves its positive minimum at some point $a \in \boldsymbol{C}$.*

Proof: Assuming that P has no root, the function $|P|$ is both continuous and positive. Once $|z|$ is sufficiently large, the expression $|c_n z^n|$ is much larger than the sum of the norms of the remaining terms of P. This means that there is some closed disk Δ so that $|P|$ is smaller at some point inside Δ than it is at all points outside Δ. Since Δ is compact, the restriction of $|P|$ to Δ has a positive minumum for some $a \in \Delta$. By the choice of Δ, the point a is where $|P|$ achieves its minimum for the whole plane. ♠

We're going to contradict this.

Lemma 0.3 *There exists some $k \geq 1$, some nonzero constant d, and some polynomial R such that*

$$P(z) = P(a) + d(z-a)^k + (z-a)^{k+1}R(z). \tag{3}$$

Proof: Consider the polynomial $Q(z) = P(z+a)$. When you expand this out, you get $Q(z) = d_0 + d_1 z + ... + d_n z^n$. Using $z - a$ in place of z,

$$P(z) = Q(z-a) = d_0 + d_1(z-a) + ... + d_n(z-a)^n.$$

Plugging in $z = a$ we see that $d_0 = P(a)$. At least one of the coefficients $d_1, ..., d_n$ is nonzero. We let k be the index of the smallest nonzero term and we set $d = d_k$. The polynomial R is just $d_{k+1} + d_{k+2}(z-a) + ...$ ♠

For comparison, we look at the simpler polynomial

$$P^*(z) = P(a) + d(z-a)^k. \tag{4}$$

Lemma 0.4 *If $|b-a|$ is sufficienly small then*

$$|P(b) - P^*(b)| < |P(a) - P^*(b)|. \tag{5}$$

Proof: We have

$$\frac{|P(b) - P^*(b)|}{|P(a) - P^*(b)|} = \frac{|R(b)|}{|d|} \times |b-a|.$$

Since a continuous function is bounded on a compact set, there is some constant M such that $\max R < M$ on the disk of radius 1 centered at a. We just take b so that $|b-a| < \min(1, |d|/M)$. ♠

Choose a small circle B centered at a so that Equation 5 holds for all $b \in B$. As z circulates once around B, the image $P^*(z)$ circulates k times around $P(a)$ on a circle centered at $P(a)$. So, if B is chosen small enough, there is some value $b \in B$ such that $P^*(b)$ lies on the line segment joining 0 to $P(a)$. We then have

$$|P(b)| \leq |P(b) - P^*(b)| + |P^*(b)| < |P(a) - P^*(b)| + |P^*(b)| = |P(a)|.$$

The first inequality is the triangle inequality. The second inequality is Equation 5. The third equality comes from the fact that $P^*(b)$ lies on the line segment joining 0 to $P(a)$. Since $|P(b)| < |P(a)|$ we have a contradiction.