Notes on topological bases

The purpose of these notes is to pin down what I said in class today about bases, because it departs from what the book says. The definition of a base for a topology given in the book is not that useful, because it does not work in the most common example, namely the case of open balls in $\boldsymbol{R}^2$. The definition I gave, which is the standard definition, is more usable.

Let X be a set. A *topological basis* for X is a collection β of subsets of X, called *basis elements*, satisfying the following axioms.

1. X is the union of elements of β. In other words, every point of X is contained in some basis element.
2. If U and V are basis elements and $p \in U \cap V$ then there is some basis element W such that $p \in W$ and $W \subset U \cap V$.

Given β, a set in X is declared *open* if X is a union of basis elements. Equivalently, a subset $U \subset X$ is declared open if every $p \in U$ has the property that there is some basis element V_p with $p \in V_p$ and $V_p \subset U$. These formulations are equivalent because

$$U = \bigcup_{p \in U} V_p.$$

Let's check the axioms for a topology.

- X is open, by Axiom 1.
- $\emptyset$ is open because technically it is the union of basis elements.
- The arbitrary union of open sets is again a union of basis elements. Hence it is open according to the definition.
- For the finite intersection property, it suffices to show that $U_1 \cap U_2$ is open when U_1 and U_2 are open. Choose some $p \in U_1 \cap U_2$. By definition, there are basis elements V_1 and V_2 so that $p \in V_1 \cap V_2$ and $V_1 \subset U_1$ and $V_2 \subset U_2$. By Axiom 2, there is some basis element W such that $p \in W \subset V_1 \cap V_2$. But then $W \subset U_1 \cap U_2$ as well. Hence $U_1 \cap U_2$ is open.

It turns out that the same topology can be induced from many different bases. Here are some examples which all give the usual topology on $\boldsymbol{R}^2$.

- The set of open disks in $\boldsymbol{R}^2$.
- The set of open squares in $\boldsymbol{R}^2$.
- The set of open squares having rational vertices.
- The set of open triangles having irrational vertices.

The way to see that these are all the same is to show that each kind of set is a union of the others. For instance, an open disk is the union of countably many open rational squares. (Hint: use a grid argument, like in class.)

Here is a really wierd example: On $\boldsymbol{R}$, the set of half-open intervals of the form $[a, b)$ is a basis. The indced topology on $\boldsymbol{R}$ is different from the usual one. To see this, I'll prove below that with the standard topology $\boldsymbol{R}$ cannot be written as the union of two nonempty disjoint open sets. On the other hand with the half-open topology, both $(-\infty, 0)$ and $[0, \infty)$ are open sets. So, in this wierd topology, $\boldsymbol{R}$ is the union of two disjoint open sets.

The following result is not really part of what I wanted to say in these notes, but it is pretty neat and I will cover it later in class too. So, just ignore this if you want.

Lemma 0.1 *$\boldsymbol{R}$ cannot be the union of two disjoint nonempty open sets.*

Proof: Suppose that $R = U \cup V$ where U and V are both open and nonempty and disjoint from each other. Pick points $u_0 \in U$ and $v_0 \in V$. Given points (u_i, v_i) consider the point w_i that is halfway between u_i and v_i. If $w_i \in U$ then set $u_{i+1} = w_i$ and $v_{i+1} = v_i$. If $w_i \in V$ then set $u_{i+1} = u_i$ and $v_{i+1} = w_i$. This construction produces points (u_n, v_n) for all n such that

- $u_n \in U$ and $v_n \in V$ for all n.
- Both u_n and v_n converge to the same point w_∞.

The point w_∞ lies either in U or V. If $w_\infty \in U$ then $v_n \in U$ for all n sufficienly large. This is a contradiction. If $w_\infty \in V$ then $u_n \in V$ for all n sufficiently large. This is also a contradiction. Since all outcomes lead to a contradiction, $\boldsymbol{R}$ cannot be written as a nontrivial union of disjoint open sets. ♠