Notes on Cauchy Sequences

The purpose of these notes is to give a clear proof of the result in class about Cauchy sequences of functions.

Let X be the set of all functions $f : [0,1] \to \boldsymbol{R}^2$. These functions need not be continuous but we only care about the continuous ones.

Given two functions $f, g \in X$ define

$$d(f,g) = \sup_{t \in [0,1]} \|g(t) - f(t)\|. \tag{1}$$

This notion of distance makes X into a metric space, though we don't need to know that for the proof.

The sequence $\{f_n\}$ is called *Cauchy* if for all $\epsilon > 0$ there is some N such that $d(f_i, f_j) \leq \epsilon$ if $i, j > N$.

Lemma 0.1 *There exists a function $g \in X$ such with the following property. For all $\epsilon > 0$ there is some N such that $d(f_n, g) \leq \epsilon$ if $n > N$.*

Proof: For each $t \in [0,1]$ the sequence $\{f_n(t)\}$ is a Cauchy sequence of real numbers. It has a limit, and we call this limit $g(t)$. This is the function g. Given $\epsilon > 0$ there is some N such that $\|f_i(t) - f_j(t)\| \leq \epsilon$ for all $i, j > N$ and for all t. But then $\|f_j(t) - g(t)\| \leq \epsilon$ for all $j > N$ and all t. The principle here is that if all the numbers of a Cauchy sequence are within ϵ of each other, then they are all within ϵ of their limit point. This principle works simultaneously for all $t \in [0,1]$. ♠

We call g the *limit* of $\{f_n\}$.

Lemma 0.2 *If the functions f_n are all continous then so is the limit g.*

Proof: We'll use the classical definition of continuity. Suppose $t_0 \in [0,1]$ and $\epsilon > 0$ are given. We want to find a $\delta < 0$ such that $|t - t_0| < \delta$ implies that $\|g(t) - g(t_0)\| < \epsilon$. By the previous lemma, there exists some n such that $d(g, f_n) < \epsilon/3$. We just need a single value of n here. Since f_n is continuous, there exists some δ such that $\|f_n(t) - f_n(t_0)\| < \epsilon/3$ if $|t - t_0| < \delta$. Taking this value, we compute

$$\|g(t) - g(t_0)\| = \|(g(t) - f_n(t) + f_n(t) - f_n(t_0) + f_n(t_0) - g(t_0)\| \leq$$

$$\|(g(t) - f_n(t)\| + \|f_n(t) - f_n(t_0)\| + \|f_n(t_0) - g(t_0)\| < 3 \times (\epsilon/3) = \epsilon.$$

That's it. ♠