Math 1410: Classification of Surfaces: The purpose of these notes is to discuss the classification of compact topological surfaces. These notes cover what I said in class, and add some details. Also, I'm going to state the final result differently than I did in class.

Triangulated Surfaces: Say that a *triangulated surface* is the identification space obtained from a finite disjoint union triangles by gluing their edges in pairs. Each edge of each triangle is labeled and given a direction. The same label is used exactly twice. The gluing has the property that the two points which are t units of the way across like-labeled edges are identified. Points in the interiors of the triangles are only equivalent to themselves; points in the interiors of edges are equivalent in pairs; otherwise many vertices could be equivalent to each other.

It is a general theorem that any compact topological surface is homeomorphic to a triangulated surface. This is quite a bit of work to prove. If you want to see a proof, look at the 1950 edition of the Annals of Mathematics journal. These notes will just deal with triangulated surfaces.

Polygon Gluing Diagrams: A related kind of surface has the following description. Start with any (solid) polygon, with an even number of edges. Then identify the edges in pairs just as for triangulated surfaces. This is called a polygon gluing diagram. Any solid polygon is homeomorphic to a solid regular polygon, so you could just work with polygon gluing diagrams based on regular polygons. This description makes these kinds of surfaces very combinatorial: You just fix a number $2n$ of sides, then pair up the edges, then choose directions for each edge.

Lemma 0.1 *Any connected triangulated surface is homeomorphic to a surface built from a polygon gluing diagram.*

Proof: Say that a *partial polygon diagram* is a solid polygon, together with a pairing of some (but possibly not all) of its edges. The diagram has a *free edge* if not all the edges are paired. If the diagram has no free edges, it defines a surface.

Let Σ be the triangulated surface. Order the N triangles in Σ. The first triangle T_1 has 3 edges, and at most 2 are glued together. So, $P_1 = T_1$ itself forms a psrtial polygon diagram whose identifications come from Σ/ Suppose that P_k is a partial polygon gluing diagram made from k of the triangles. If

$k < N$ then P_k must have a free edge. Otherwise P_k defines a compact surface and none of the other triangles can attach to P_k. This contradicts the connectivity of Σ.

So, P_k has a free edge. This free edge is paired with some edge that belongs to a triangle T not on the list of those comprising P_k. Let P_{k+1} be the larger polygon obtained by attaching T to P_k along the relevant edge, and according to the pairings for Σ. Adjust the shape of T if necessary to guarantee that P_{k+1} is still embedded. If some other edge of T is paired with some edge of P_k, then add this to the data for the diagram. Otherwise, do nothing. By construction P_{k+1} is a partial gluing diagram that just uses the pairings from Σ.

By induction, the partial gluing diagram P_N exists. This diagram cannot have a free edge, because every edge of every triangle of Σ is paired with some other edge. So, the gluing diagram of P_N is complete: the edges are all paired. The pairings for P_N are the same as some of those for Σ, and the remaining pairings from Σ correspond to edges in the interior of P_N. Therefore, the identification space obtained by gluing together the edges on the boundary of P_N is homeomorphic to Σ. ♠

Complexity: Suppose now that Σ is a polygon gluing diagram. We're going to prove that Σ is either the sphere, or homeomorphic to a connected sum of tori, Klein bottles, and projective planes. Define the *complexity* of Σ to be the minimum number of sides in and polygon gluind diagram that produces a surface homeomorphic to Σ. For instance, the torus, sphere, Klein bottle, and projective plane all have complexity 4. The proof is going to be induction on the complexity. So, suppose that Σ is a surface with the smallest complexity for which we have not yet proved the result.

Crossing Pairs: Let Σ be a surface as above and let P be a minimal complexity gluing diagram for Σ. Say that a *crossing pair* for Σ is a collection of 4 edges $A_1, A_2.B_1, B_2$ such that $A_1 \cup A_2$ separates $B_1 \cup B_2$ on ∂P. In other words, the edges are interlaced. Here we mean that A_1, A_2 are paired and B_1, B_2 are paired.

Suppose Σ has a crossing pair. We can move the gluing diagram by a homeomorphism so that A_1, A_2, B_1, B_2 are contained in the 4 sides of a square and all of P is contained in the solid square. This is shown in Part 1 of Figure 1.

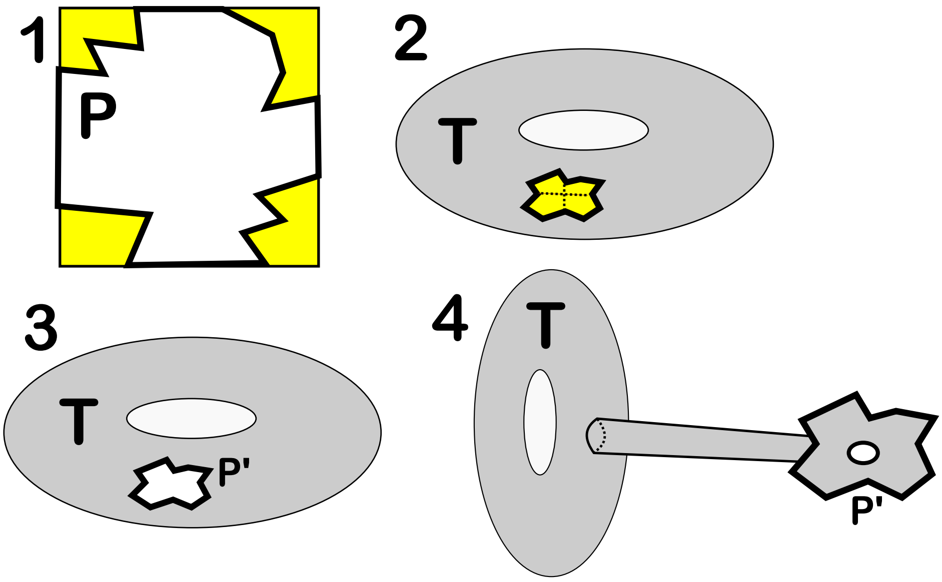

Figure 1: The induction step

Think of the square as the union of P and some yellow pieces that will go along as guides for the construction. Pair the sides of the square according to the labels on A_1, A_2, B_1, B_2. Now glue the square together. This will produce either a torus, a Klein bottle, or a projective plane. Figure 1 shows the torus case. Part 2 of Figure 1 shows the torus T together with the images of the yellow pieces.

Next, delete the yellow pieces and recall the pairings on the sides of the resulting polygonal hole P' in T. This is shown in Part 3 of Figure 1. Finally, imagine that T is made of rubber, and you pull P' away from T. In part 4 of Figure 1, the polygon P' is joined back to T by a long tube that flares out at the end. From this picture, we recognize that Σ is the connected sum of a torus with a simpler surface Σ' whose gluing diagram is given by the solid polygon whose boundary is P'.

In short, Σ is the connected sum of a torus with a surface Σ' of lower complexity. Hence, y induction, the main claim holds for Σ. Even though we just dealt with the torus case, the same argument would work in the Klein bottle or projective plane case: We would see that Σ is the connected sum of Σ' with either a Klein bottle or a projective plane.

Adjacent Pairs, Part 1: Let Σ and P be as above. If P has no crossing pair, then P has a pair of adjacent edges that are glued together. The most elementary way to see this is to consider the pair of edges A_1, A_2 that are paired together and as close together as possible. If these edges are not adjacent, then there is some edge B_1 between A_1 and A_2. But B_1 must be paired with an edge B_2 between A_1 and A_2 and on the same side, because there are no crossing pairs. But then (B_1, B_2) are closer together and we have a contradiction.

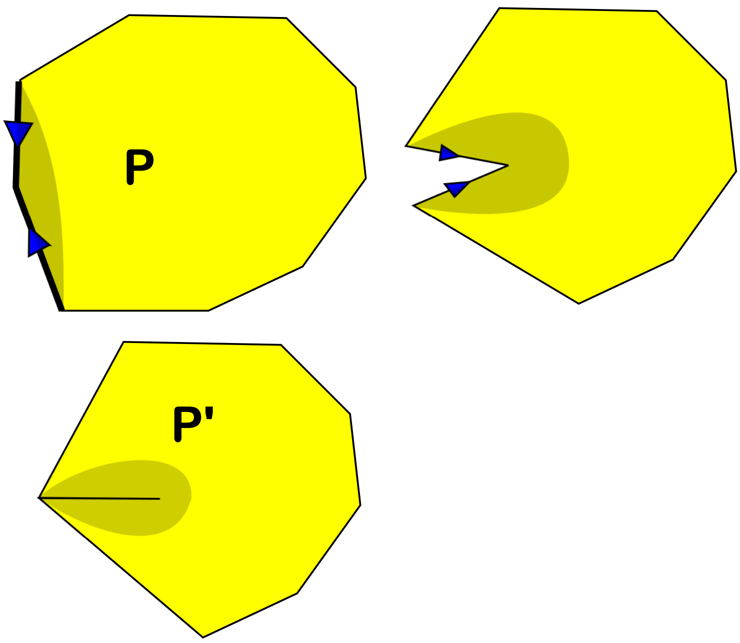

Figure 2: Oppositely oriented adjacent edges.

So, let A_1 and A_2 be adjacent edges that are glued together. Suppose first that the directions on A_1 and A_2 are opposite, as shown in Figure 2. Then we can re-draw P in such a way that A_1 and A_2 are nearly identified. Then we make the gluing we see that the resulting surface is homeomorphic to the surface based on the gluing diagram P' that comes from omitting A_1 and A_2. In this case, we have a contradiction. The polygon P could not have been a minimal gluing diagram because P' has 2 fewer edges and gives the same surface.

Before we move on to the next section I want to explain something more about Figure 2. I have shaded a neighborhood of the two edges to illustrate the fact that cutting out this neighborhood just amounts to cutting out a disk from the final surface.

Adjacent Pairs, part 2: Suppose that the gluing diagram P has two adjacent pairs that have the same direction. Let T be the triangle made from these two edges and from the third edge that joins the non-adjacent vertices of these two edges. The triangle T is shown in red in Figure 3. The identification space $T/\sim$ is a Mobius band. This is most easily seen by re-cutting T along a segment that bisects T

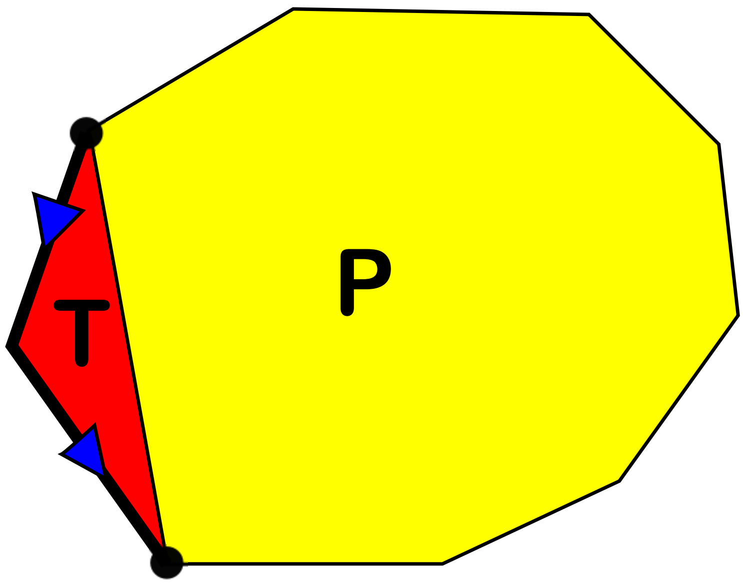

Figure 3: Same-oriented adjacent edges.

The identification space $(P-T)/\sim$, which is the yellow part of Figure 3 modulo the equivalence relation, is homeomorphic to the result of cutting a disk out of a lower complexity surface. The best way to see this is to let T' be the modification of T obtained by switching the directions of one of the edges. Then $(P-T) \cup T'$ is the same kind of gluing diagram we considered in the previous section, and it produces a surface Σ' of lower complexity than Σ. But then we recognize $(P-T)/\sim$ as the result of cutting a disk out of Σ'. This is why we had the discussion at the end of the last section.

The total space $P/\sim$ is therefore obtained by cutting out a disk from Σ' and gluing a Mobius band along the boundary of that disk. Since the projective plane minus a disk is homeomorphic to a Mobius band, we see that Σ is the connected sum of Σ' and a Mobius band. Therefore, by induction, Σ is either a sphere, or the connected sum of tori, Klein bottles, and projective planes.

Endgame: Now we know that any triangulated surface is either the sphere or homeomorphic to the connected sum of tori, Klein bottles, and projective planes. We're going to simplify the picture in this section.

First of all, the Klein bottle is the connected sum of two projective planes. So, we can eliminate any Klein bottles from the description: Any triangulated surface is homeomorphic to either the sphere or a connected sum of tori and projective planes.

Second of all, the connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes. So, if a surface is homeomorphic to a connected sum of both tori and projective planes, then it is also homeomorphic to the connected sum of only projective planes.

In summary, any triangulated surface is homeomorphic to one of three kinds of surfaces:

- The sphere.
- A connected sum of tori.
- A connected sum of projective planes.

Now I'll sketch the proof that no two of these surfaces is homeomorphic to each other. This shows that each triangulated surface is homeomorphic to *exactly* one surface on the list.

The sphere is the only simply connected example on the list. So, it is not homeomorphic to any of the others. The torus surfaces do not contain any embedded Mobius bands whereas the projective plane surfaces all do. So, no surface of the second kind of homeomorphic to a surface of the third kind.

If we remove one point from the connected sum of n tori, the resulting space is has the same homotopy type as a bouquet of $2n$ circles. (This is most easily seen by noting that the connected sum of n tori has a description as the surface obtained by gluing opposite sides of a $4n$-gon.) The fundamental group of this space is the free group on $2n$ generators, which is not isomorphic to the free group on $2m$ generators when $m \neq n$. Hence two different surfaces of the same type are never homeomorphic. The same argument works almost word for word for surfaces of the third type.