Math1410: Crash Course on Forms and Cohomology: The purpose of these notes is to discuss differential forms, Stokes’ Theorem, and cohomology. These notes will be somewhat sketchy.

**Tensors:** Let $V$ be a real vector space. It is convenient just to take $V = \mathbb{R}^n$. Let $V^k$ denote the set of $k$-tuples of vectors in $V$. A point in $V^k$ is a $k$-tuple $(V_1, ..., V_k)$. A $k$-tensor is a map $T: V^k \to \mathbb{R}$ such that

$$T(..., aV_i + bW_i, ...) = aT(..., V_i, ...) + bT(..., W_i, ...).$$

That is, $T$ is linear in each slot. The dot product is a classic example of a 2-tensor.

A tensor is called *alternating* if

$$T(..., V_i, ..., V_j, ...) = -T(..., V_j, ..., V_i, ...)$$

for all $i, j$. The determinant is an example of an alternating $n$-tensor on $\mathbb{R}^n$.

Here is a more general example of an alternating $k$-tensor on $\mathbb{R}^n$ for $k \leq n$. Let $a_1, ..., a_k$ be some sequence of integers between 1 and $n$. Given vectors $V_1, ..., V_k$, define

$$T(V_1, ..., V_k) = \det \begin{bmatrix} V_{1a_1} & \cdots & V_{1a_k} \\ \vdots \\ V_{ka_1} & \cdots & V_{ka_k} \end{bmatrix}.$$ 

Here $V_i = (V_{i1}, ..., V_{in})$, etc. This tensor is denoted $dx_{a_1} \wedge \cdots \wedge dx_{a_k}$. Note that this tensor is zero if there are repeated indices. Note also that the tensor switches signs if you switch two indices. For instance, when $k = 3$ and $n = 5$ we have

$$dx_1 \wedge dx_3 \wedge dx_4 = -dx_3 \wedge dx_1 \wedge dx_4.$$ 

If turns out that every alternating $k$-tensor is a linear combination of the examples given. Therefore, the vector space of alternating $k$-tensors has dimension $n$ choose $k$. Alternate notation: Given a $k$-tuple $I = (a_1, ..., a_k)$, we let $dx_I$ be the form mentioned above.

**Differential Forms:** Let $U$ be an open subset of $\mathbb{R}^n$. A *differential $k$-form* is a smoothly varying choice of alternating $k$-tensor for each point of
Given that we have a basis for the vector space of alternating $k$-tensors, we can say more concretely that a differential $k$-form is a sum of the form

$$\alpha = \sum f_I dx_I, \quad (1)$$

each $f_I$ is a smooth (i.e. infinitely differentiable) function on $U$. Here are some examples, on $\mathbb{R}^3$:

- The 0-forms are just functions.
- The 1-forms look like $A_1 dx_1 + A_2 dx_2 + A_3 dx_3$ where $A_1, A_2, A_3$ are functions.
- The 2-forms look like $A_1 dx_1 \wedge dx_2 + A_2 dx_1 \wedge dx_3 + A_3 dx_2 \wedge dx_3$ where $A_1, A_2, A_3$ are functions.
- The 3-forms look like $A dx_1 \wedge dx_2 \wedge dx_3$ where $A$ is a function.

Note that we can interpret a function either as a 0-form or a 3-form. Likewise we can interpret the vector field $(A_1, A_2, A_3)$ either as a 1-form or a 2-form.

To give a more exotic example, the 2-forms on $\mathbb{R}^4$ look like

$$\sum_{i=1}^{4} \sum_{j=i+1}^{4} a_{ij} dx_i \wedge dx_j.$$  

There are 6 summands, and each $a_{ij}$ is a function.

**The $d$ Operator:** When $f$ is a function, we define

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$  

In other words, $df$ is a 1-form. For the general $k$ form defined in Equation 1, we have

$$d\alpha = \sum df_I \wedge dx_I. \quad (2)$$

Consider the 4 possibilities in $\mathbb{R}^3$:

- Suppose we start with a function $f$, compute $df$, then interpret the result as a vector field. The result is just the gradient of $f$. 


• Suppose we start with a vector field, interpret it as a 1 form $f$, then re-interpret $df$ as a vector field. Then we get the curl. Here is the main part of the calculation:

$$d \sum_{i=1}^{3} A_i \, dx_i = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial A_i}{\partial x_j} \, dx_j \wedge dx_i = \left( \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} \right) dx_1 \wedge dx_2 + \left( \frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2} \right) dx_2 \wedge dx_3 + \left( \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} \right) dx_3 \wedge dx_1.$$  

• Suppose we start with a vector field, interpret it as a 2-form, apply $d$, then interpret the result as a function. Then we are computing the divergence.

So, for differential forms in $\mathbb{R}^3$ the $d$-operator unifies all the basic operations from vector calculus: gradient, curl, and divergence.

**Cohomology:** In vector calculus you learn the basic facts that

$$\text{curl} \circ \text{grad} = 0, \quad \text{div} \circ \text{curl} = 0.$$  

Using the interpretations above, both these statements just say that $d \circ d = 0$. This is true in general. Here is the calculation, for the form $\alpha$ in Equation 1:

$$d(\alpha) = d \left( \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \wedge dx_i \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f_i}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_j.$$  

But this whole thing is zero because

$$\frac{\partial^2 f_i}{\partial x_i \partial x_j} dx_i \wedge dx_j - \frac{\partial^2 f_i}{\partial x_i \partial x_j} dx_j \wedge dx_i - \frac{\partial^2 f_i}{\partial x_j \partial x_i} dx_j \wedge dx_i.$$  

That is, the terms in the sum cancel in pairs.

A $k$-form $\alpha$ is called **closed** if $d\alpha = 0$. A $k$-form $\alpha$ is called **exact** if $\alpha = d\beta$. The calculation above says that “exact implies closed”. As an alternate terminology, exact forms are called **coboundaries** and closed forms are called **cocycles**. This alternate terminology is supposed to line up with the language of homology.

Let $M$ be some open subset of $\mathbb{R}^n$. We define:

• $C^k(M)$ is the vector space of $k$-forms on $M$.  

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\[ Z^k(M) \text{ is the vector space of closed } k\text{-forms on } M. \]
\[ B^k(M) \text{ is the vector space of exact } k\text{-forms on } M. \]

The fact that \( d \circ d = 0 \) means that \( B^k(M) \subset Z^k(M) \). The quotient group

\[ H^k(M) = Z^k(M)/B^k(M) \quad (3) \]

is known as the \( k\)-th deRham cohomology group of \( M \).

**Connection to Simplicial Homology:** Here I’ll explain a special case of the connection between the deRham cohomology defined above and simplicial homology. Suppose that \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is a smooth function. Suppose also that there are no points \( p \in \mathbb{R}^n \) where \( f(p) = 0 \) and \( \nabla f(p) = 0 \). In other words, the function and its gradient cannot vanish at the same point.

In this situation, \( f^{-1}(0) \) is a well defined manifold of dimension \( n - 1 \). The tangent space at some point \( p \) is the subspace perpendicular to \( \nabla p \).

We can look at \( \Sigma = f^{-1}(0) \) in two ways.

- We can triangulate \( \Sigma \) and treat it as a simplicial complex. Then we can compute the simplicial homology \( H_i(\Sigma, \mathbb{R}) \). What we do is take real linear combinations of simplices and then take cycles mod boundaries. Here \( H_k(M, \mathbb{R}) \) is not just a group but also a vector space.

- We can consider the open set \( M_\epsilon = f^{-1}(-\epsilon, \epsilon) \) for some very small \( \epsilon \).

The space \( M_\epsilon \) is an open set in \( \mathbb{R}^n \) which is a kind of thickening of \( \Sigma \).

For small \( \epsilon \), the space \( M_\epsilon \) is homeomorphic to \( \Sigma \times (-1, 1) \). We can then compute the cohomology groups \( H^k(M_\epsilon) \).

Here is the punchline. For sufficiently small \( \epsilon \), the groups \( H_k(\Sigma, \mathbb{R}) \) and \( H^k(M) \) are isomorphic vector spaces. This is a special case of the famous *De Rham Isomorphism Theorem*. One take-away from this result is that the basic notions in vector fields: incompressible vector fields, irrotational vector fields, conservative potentials, etc., are all closely related to homology.

**Integration and Stokes Theorem:** The isomorphism theorem discussed above has close connections to Stokes’ Theorem. The basic fact is that a \( k\)-form can be integrated over a \( k\)-dimensional manifold.

Suppose first that \( T \) is an alternating \( k\)-tensor and \( \Delta \) is a \( k\)-dimensional simplex. If we label the vertices of \( \Delta \) as \( \Delta(0), ..., \Delta(k) \) then we get the
When $T$ is alternating this number only depends very mildly on the labeling. If we change the labeling by an even permutation, then the answer does not change. So, in short, an alternating $k$-tensor assigns a number to an oriented $k$-simplex.

Now suppose that we have some oriented $k$-dimensional manifold $M$ in $\mathbb{R}^n$ and some $k$-form $\omega$. We can triangulate $M$ into small simplicess, say $M = \Delta_1 \cup \ldots \cup \Delta_\ell$, and we can arrange that the orientations of the simplices are chosen so that the union of simplices is a $k$-chain. (This last arrangement requires $M$ to be oriented.)

We can then define the sum

$$
\sum_{i=1}^\ell T_i(\Delta_i),
$$

where $T_i$ is the tensor we get by evaluating $\omega$ at some point in $\Delta_i$ (like the center of mass.) You should think of this sum as like a Riemann sum from the theory of integration. Letting the mesh size of the triangulation go to 0 and taking a limit, we can get a well defined answer, which we call

$$
\int_M \omega,
$$

the integral of $\omega$ over $M$.

Suppose finally that $\Omega$ is a $(k+1)$-dimensional manifold with a boundary $\partial \Omega$, and $\omega$ is a $k$-form defined on an open neighborhood of $\Omega$. The form $d\omega$ is a $(k+1)$ form and it makes sense to integrate it on $\Omega$ whereas $\omega$ is a $k$-form and it makes sense to integrate $\omega$ on $\partial \Omega$. Here is the general version of Stokes’ Theorem:

$$
\int_\Omega d\omega = \int_{\partial \Omega} \omega. \quad (4)
$$

This one result, when suitably interpreted, encompasses all the results from vector calculus – Green’s Theorem, the Divergence Theorem, Gauss’s law, Stokes’ Theorem.

At the same time, Equation 4 hints at a direction between cohomology and homology by relating the operation of taking boundaries (homology) with the $d$-operation (cohomology).