Math1410: Crash Course on Forms and Cohomology: The purpose of these notes is to discuss differential forms, Stokes' Theorem, and cohomology. These notes will be somewhat sketchy.

Tensors: Let V be a real vector space. It is convenient just to take $V = \boldsymbol{R}^n$. Let V^k denote the set of k-tuples of vectors in V. A point in V^k is a k-tuple $(V_1, ..., V_k)$. A *k-tensor* is a map $T : V^k \to \boldsymbol{R}$ such that

$$T(..., aV_i + bW_i, ...) = aT(..., V_i, ...) + bT(..., W_i, ...).$$

That is, T is linear in each slot. The dot product is a classic example of a 2-tensor.

A tensor is called *alternating* if

$$T(..., V_i, ..., V_j, ...) = -T(..., V_j, ..., V_i, ...)$$

for all i, j. The determinant is an example of an alternating n-tensor on $\boldsymbol{R}^n$.

Here is a more general example of an alternating k-tensor on $\boldsymbol{R}^n$ for $k \leq n$. Let $a_1, ..., a_k$ be some sequence of integers between 1 and n. Given vectors $V_1, ..., V_k$, define

$$T(V_1, ..., V_k) = \det \begin{bmatrix} V_{1a_1} & ... & V_{1,a_k} \\ ... & & \\ V_{ka_1} & ... & v_{k,a_k} \end{bmatrix}.$$

Here $V_1 = (V_{11}, ..., V_{1n})$, etc. This tensor is denoted $dx_{a_1} \wedge ... \wedge dx_{a_k}$. Note that this tensor is zero if there are repeated indices. Note also that the tensor switches signs if you switch two indices. For instance, when $k = 3$ and $n = 5$ we have

$$dx_1 \wedge dx_3 \wedge dx_4 = -dx_3 \wedge dx_1 \wedge dx_4.$$

If turns out that every alternating k-tensor is a linear combination of the examples given. Therefore, the vector space of alternating k-tensors has dimension n choose k. Alternate notation: Given a k-tuple $I = (a_1, ..., a_k)$, we let dx_I be the form mentioned above.

Differential Forms: Let U be an open subset of $\boldsymbol{R}^n$. A *differential k-form* is a smoothly varying choice of alternating k-tensor for each point of

U. Given that we have a basis for the vector space of alternating k-tensors, we can say more concretely that a differential k-form is a sum of the form

$$\alpha = \sum f_I dx_I, \tag{1}$$

each f_I is a smooth (i.e. infinitely differentiable) function on U. Here are some examples, on $\boldsymbol{R}^3$:

- The 0-forms are just functions.
- the 1-forms look like $A_1 dx_1 + A_2 dx_2 + A_3 dx_3$ where A_1, A_2, A_3 are functions.
- The 2-forms look like $A_1 dx_1 \wedge dx_2 + A_2 dx_1 \wedge dx_3 + A_3 dx_2 \wedge dx_3$ where A_1, A_2, A_3 are functions.
- The 3-forms look like $A dx_1 \wedge dx_2 \wedge dx_3$ where A is a function.

Note that we can interpret a function either as a 0-form or a 3-form. Likewise we can interpret the vector field (A_1, A_2, A_3) either as a 1-form or a 2-form.

To give a more exotic example, the 2-forms on $\boldsymbol{R}^4$ look like

$$\sum_{i=1}^{4} \sum_{j=i+1}^{4} a_{ij} dx_i \wedge dx_j.$$

There are 6 summands, and each a_{ij} is a function.

The d Operator: When f is a function, we define

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

In other words, df is a 1-form. For the general k form defined in Equation 1, we have

$$d\alpha = \sum df_I \wedge dx_I. \tag{2}$$

Consider the 4 possibilities in $\boldsymbol{R}^3$.

- Suppose we start with a function f, compute df, then interpret the result as a vector field. The result is just the gradient of f.

- Suppose we start with a vector field, interpret it as a 1 form f, then re-interpret df as a vector field. Then we get get the curl. Here is the main part of the calculation.

$$d\sum_{i=1}^{3} A_i dx_i = \sum_{i=1}^{3} dA_i \wedge dx_i = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial A_i}{\partial x_j} dx_j \wedge dx_i =$$

$$\Big(\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1}\Big) dx_1 \wedge dx_2 + \Big(\frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2}\Big) dx_2 \wedge dx_3 + \Big(\frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3}\Big) dx_3 \wedge dx_1.$$

- Suppose we start with a vector field, interpret it as a 2-form, apply d, then interpret the result as a function. Then we are computing the divergence.

So, for differential forms in $\boldsymbol{R}^3$ the d-operator unifies all the basic operations from vector calculus: gradient, curl, and divergence.

Cohomology: In vector calculus you learn the basic facts that

$$\text{curl} \circ \text{grad} = 0, \qquad \text{div} \circ \text{curl} = 0.$$

Using the interpretations above, both these statements just say that $d \circ d = 0$. This is true in general. Here is the calculation, for the form α in Equation 1:

$$d(d\alpha) = d\Big(\sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} \wedge dx_I\Big) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f_I}{\partial x_i \partial x_i} dx_i \wedge dx_j \wedge dx_I.$$

But this whole thing is zero because

$$\frac{\partial^2 f_I}{\partial x_i \partial x_i} dx_i \wedge dx_j - \frac{\partial^2 f_I}{\partial x_i \partial x_j} dx_j \wedge dx_i - \frac{\partial^2 f_I}{\partial x_j \partial x_i} dx_j \wedge dx_i.$$

That is, the terms in the sum cancel in pairs.

A k-form α is called *closed* if $d\alpha = 0$. A k-form α is called *exact* if $\alpha = d\beta$. The calculation above says that "exact implies closed". As an alternate terminology, exact forms are called *coboundaries* and closed forms are called *cocycles*. This alternate terminology is supposed to line up with the language of homology.

Let M be some open subset of $\boldsymbol{R}^n$. We define:

- $C^k(M)$ is the vector space of k-forms on M.

- $Z^k(M)$ is the vector space of closed k-forms on M.
- $B^k(M)$ is the vector space of exact k-forms on M.

The fact that $d \circ d = 0$ means that $B^k(M) \subset Z^k(M)$. The quotient group

$$H^k(M) = Z^k(M)/B^k(M) \tag{3}$$

is known as the *k-th deRham cohomology group* of M.

Connection to Simplicial Homology: Here I'll explain a special case of the connection between the deRham cohomology defined above and simplicial homology. Suppose that $f : \boldsymbol{R}^n \to \boldsymbol{R}$ is a smooth function. Suppose also that there are no points $p \in \boldsymbol{R}^n$ where $f(p) = 0$ and $\nabla f(p) = 0$. In other words, the function and its gradient cannot vanish at the same point. In this situation, $f^{-1}(0)$ is a well defined manifold of dimension $n - 1$. The tangent space at some point p is the subspace perpendicular to ∇p.

We can look at $\Sigma = f^{-1}(0)$ in two ways.

- We can triangulate Σ and treat it as a simplicial complex. Then we can compute the simplicial homology $H_i(\Sigma, \boldsymbol{R})$. What we do is take *real* linear combinations of simplices and then take cycles mod boundaries. Here $H_k(M, \boldsymbol{R})$ is not just a group but also a vector space.
- We can consider the open set $M_\epsilon = f^{-1}(-\epsilon, \epsilon)$ for some very small ϵ. The space M_ϵ is an open set in $\boldsymbol{R}^n$ which is a kind of thickening of Σ. For small ϵ, the space M_ϵ is homeomorphic to $\Sigma \times (-1, 1)$. We can then compute the cohomology groups $H^k(M_\epsilon)$.

Here is the punchline. For sufficiently small ϵ, the groups $H_k(\Sigma, \boldsymbol{R})$ and $H^k(M)$ are isomorphic vector spaces. This is a special case of the famous *De Rham Isomorphism Theorem*. One take-away from this result is that the basic notions in vector fields: incompressible vector fields, irrotational vector fields, conservative potentials, etc., are all closely related to homology.

Integration and Stokes Theorem: The isomorphism theorem discussed above has close connections to Stokes' Theorem. The basic fact is that a k-form can be integrated over a k-dimensional manifold.

Suppose first that T is an alternating k-tensor and Δ is a k-dimensional simplex. If we label the vertices of Δ as $\Delta(0), ..., \Delta(k)$ then we get the number

$$T(\Delta) = T(V_1, ..., V_k), \qquad V_j = \Delta(j) - \Delta(0).$$

When T is alternating this number only depends very mildly on the labeling. If we change the labeling by an even permutation, then the answer does not change. So, in short, an alternating k-tensor assigns a number to an oriented k-simplex.

Now suppose that we have some oriented k-dimensional manifold M in $\boldsymbol{R}^n$ and some k-form ω. We can triangulate M into small simpliciess, say $M = \Delta_1 \cup ... \cup \Delta_\ell$, and we can arrange that the orientations of the simplices are chosen so that the union of simplices is a k-chain. (This last arrangement requires M to be oriented.)

We can then define the sum

$$\sum_{i=1}^{\ell} T_i(\Delta_i),$$

where T_i is the tensor we get by evaluating ω at some point in Δ_i (like the center of mass.) You should think of this sum as like a Riemann sum from the theory of integration. Letting the mesh size of the triangulation go to 0 and taking a limit, we can get a well defined answer, which we call

$$\int_M \omega,$$

the integral of ω over M.

Suppose finally that Ω is a $(k+1)$-dimensional manifold with a boundary $\partial\omega$, and ω is a k-form defined on an open neighborhood of Ω. The form $d\omega$ is a $(k+1)$ form and it makes sense to integrate it on Ω whereas ω is a k-form and it makes sense to integrate ω on $\partial\Omega$. Here is the general version of Stokes' Theorem:

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega. \qquad (4)$$

This one result, when suitably interpreted, encompasses all the results from vector calculus – Green's Theorem, the Divergence Theorem, Gauss's law, Stokes Theorem.

At the same time, Equation 4 hints at a direction between cohomology and homology by relating the operation of taking boundaries (homology) with the d-operation (cohomology).