Math 1410: HW Assignment 4:

1: Let X be a solid tetrahedron that has been divided into smaller tetrahedra in such a way that any two sub-tetrahedra A and B of the subdivision are either disjoint, share a common point, share a common edge, or share a common face. Suppose that the vertices of the subdivision have been labeled by integers $1, 2, 3, 4$ in such a way that on the jth face of the boundary, ∂X, there are no j labels. Prove that there is some sub-tetrahedron of the subdivision whose vertices get all 4 labels.

2. Let X be a solid tetrahedron. Prove that there is no continuous map $f : X \to \partial X$ which is the identity on ∂X. Hint: Prob. 1 and compactness.

3. Use Problem 2 to prove the 3-dimensional Brouwer fixed point theorem. Any continuous map from a ball to itself has a fixed point. You can use the fact that a solid tetrahedron is homeomorphic to a ball.

4. Suppose that T is a solid triangle that has been triangulated in such a way that any two sub-triangles A and B of T are either disjoint, share a vertex, or share an edge. Suppose that T has been labeled by integers $1, 2, 3$ in such a way that, as you go clockwise around the boundary, there are more instances when a 1 label changes to a 2 than there are instances where a 2 label changes to a 1. Prove that there is some sub-triangle of the tetrahedron whose vertices get all 3 labels. Hint: Compare the number of oriented 12-flags with the number of oriented 21-flags.

5. Use Problem 4 to prove the following generalization of the no-retraction Theorem: Suppose that Δ is the unit disk. Prove that there is no continuous map $f : \Delta \to \partial\Delta$ such that the restriction of f to $\partial\Delta$ agrees with the map $f(z) = z^2$ (using complex notation). Incidentally, this result also works for the map $z \to z^n$ as long as $n \neq 0$.

Extra Credit: Take the same set-up in problem 4 and show that the map in question does not exist under the following weaker assumption: The restriction of f to $\partial\Delta$ has the property that $f(z)$ and z^n are never opposite points on the unit circle. Hint: Assume f exists and try to tweak it so that you get a map contradicting Problem 5. Next: Use the result you just proved to establish the Fundamental Theorem of Algebra.