Math 1410: HW Assignment 5:

1: a. Prove that every Alexander cycle contains a closed polygonal loop.
b. Let C be an Alexander cycle and let $L \subset C$ be a closed polygonal loop. Prove that the closure of $C - L$ is an Alexander cycle.
c. Let C_1 and C_2 be two Alexander cycles which have a transverse intersection. Prove that $C_1 \cap C_2$ is an even number of points.

2: Two solid rectangles are *almost disjoint* if they have disjoint interiors. Prove that every Alexander cycle is the boundary of a finite union of almost disjoint rectangles.

3: A *space polygon* is a finite union of segments $I_1, ..., I_n$ in $\boldsymbol{R}^3$ such that consecutive segments abut (i.e. share a common endpoint) and non-consecutive segments are disjoint. Also, the first endpoint of I_1 coincides with the last endpoint of I_n. Prove that the complement of a space polygon is connected.

4: Let S be a space polygon and let $\epsilon > 0$ be given. Prove that there exists another space polygon S' whose sides are parallel to the coordinate axes such that every point of S is within ϵ of S' and *vice versa*. In other words, you can approximate space polygons by "rectilinear space polygons".

5: This problem is phrased in such a way as to avoid the need to bring in measure theory. Say that a set in the plane is *ϵ-small* if it is a countable union $T_1 \cup T_2 \cup T_3 ...$ of triangles such that

$$\sum_{i=1}^{\infty} \text{area}(T_i) < \epsilon.$$

Prove that there exists a Jordan loop contained in the unit square whose complement is ϵ-small. Hint: look at Figure 1.

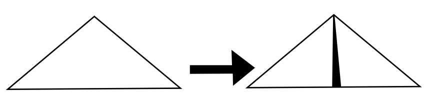

Figure 1: Suggested subdivision operation

In terms of measure theory, this result says that the unit square (which has measure 1) contains Jordan loops having measure at least $1 - \epsilon$.

6: A subset of $\boldsymbol{R}^3$ is called a *triangulated surface* if it is a finite union of triangles satisfying the following axioms:

- Every two triangles in the union are either disjoint, or intersect in a common edge, or intersect in a common vertex.
- If T is a triangle in the collection and e is an edge of T, then there is a unique second triangle T' in the collection such that e is also an edge of T'. In this case we write $T \sim T'$.
- If v is a vertex of some triangle T_1 in the collection, then there are triangles $T_2, ..., T_k$ such that $T_1 \sim T_2 \sim ... \sim T_k \sim T_1$, and the common intersection is v. In other words, these triangles fit around v like slices of a pizza.

For instance, the regular tetrahedron and the regular octahedron are triangulated surfaces. Prove that, for any $\epsilon > 0$, there is a triangulated surface within ϵ of the round sphere in $\boldsymbol{R}^3$. In other words, every point of the surface is within ϵ of the sphere and *vice versa*. So, you can approximate the sphere by triangulated surfaces in the same way that you can approximate Jordan loops by polygons.

Bonus Problem: Make a definition of transverse intersections for triangulated surfaces and then prove that if two triangulated surfaces have transverse intersection then their intersection is a finite union of space polygons.