**Math 1410: Classic Examples of Manifolds:**

The purpose of these notes is to explain some classic examples of manifolds. This won't be on the exam! All these examples are compact Hausdorff spaces which are topological manifolds. The examples also have a number of additional structures associated to them which I am not going to discuss in these notes. For instance, they are usually considered as smooth manifolds (whatever that means).

**A General Principle:** Let $X$ be a topological space and let $\sim$ be an equivalence on $X$. Call a subset $A \subset X$ *full* (with respect to the equivalence relation) if every equivalence class intersects $A$. We can treat $A$ as a topological space by giving it the subspace topology, and then we can consider $A/\sim$.

**Lemma 0.1** *If $A$ is compact and $A/\sim$ is Hausdorff Then $X/\sim$ and $A/\sim$ are homeomorphic.*

**Proof:** The basic idea is to shoehorn this result into the one thing we know: A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

Let $\phi : X \to X/\sim$ be the quotient map. The identity map $\iota : A \to X$ respects the equivalence relation and gives a map from $A/\sim$ to $X/\sim$. The map $I$ is surjective because $A$ intersects every equivalence class. The map $I$ is injective by definition of the equivalence relation on $A$: Two elements in $A$ are equivalent if and only if they are equivalent in $X$. So, $I$ is a bijection.

Now we show that $I$ is continuous. The map $I$ is induced by the map

$$\phi\iota : A \to X/\sim .$$

So, we just have to show that $\phi\iota$ (the composition) is continuous. Choose some open $U \in X/\sim$. By definition $\phi^{-1}(U)$ is open in $X$. But

$$(\phi\iota)^{-1}(U) = \phi^{-1}(U) \cap A,$$

which is open by definition of the subspace topology. Hence $I$ is continuous.

Since $A$ is compact in $X$ and $\phi : X \to X/\sim$ is continuous, $\phi(A)$ is compact. But $\phi(A) = X/\sim$ because every equivalence class intersects $A$. Hence

$X/\sim$ is compact. Now we know that $I$ is a continuous bijection from a compact space to a Hausdorff space. That means that $I$ is a homeomorphism. ♠

**Real Projective Space:** $\boldsymbol{RP}^n$ is defined to be $X/\sim$ where $X$ is the space of nonzero vectors in $\boldsymbol{R}^{n+1}$ and $v \sim w$ if and only if $v$ and $w$ are multiples of each other. Equivalently, $v$ and $w$ are equivalent if and only if they are contained in the same line through the origin. This $\boldsymbol{RP}^n$ is the space of lines through the origin in $\boldsymbol{R}^{n+1}$.

To understand our example, we use the general principle above. Let $A$ be the unit sphere in $\boldsymbol{R}^{n+1}$. The set $A$ is full because every nonzero vector is equivalent to a unit vector. Also, $A$ is compact by the Heine-Borel theorem: it is closed and bounded. Finally the relation on $A$ is just that $v \sim \pm v$. The open sets in $A/\sim$ are quotients of open sets $U$ of $A$ which are invariant under the antipodal map. That is, $v \in U$ if and only if $-v \in U$. If $[v]$ and $[w]$ are two points in $A/\sim$ then we can place $\pm u$ and $\pm v$ inside small and disjoint symmetric open sets. Each symmetric set is just a union of two small balls centered at the relevant points. Hence $A/\sim$ is Hausdorff. The general principle tells us that $\boldsymbol{RP}^n$ is homeomorphic to $A/\sim$, which is described in words as the sphere modulo the antipodal map.

As a third definition, let $B \subset A$ denote the upper hemisphere of $A$, including the equator. The space $B$ is again compact, and a similar argument to the one above shows that $B/\sim$ is Hausdorff. Hence $A/\sim$ and $B/\sim$ are homeomorphic. The equivalence relation on $B$ is as follows: Each point in the interior of $B$ is its own equivalence class and the equivalence classes on the equator, $\partial B$, look like $\pm v$ for unit vectors $v$. We can identify the upper hemisphere of $A$ with the unit ball in $\boldsymbol{R}^n$. So, we can equally well describe $\boldsymbol{RP}^n$ is the solid $n$-ball with antipodal points in its boundary identified. This last definition reveals that $\boldsymbol{RP}^n$ is the union of a space homeomorphic to $\boldsymbol{R}^n$ and $\boldsymbol{RP}^{n-1}$. Abusing notation somewhat, we can write $\boldsymbol{RP}^n = \boldsymbol{R}^n \cup \boldsymbol{RP}^{n-1}$.

Here is another way to think about this union. $\boldsymbol{R}^n$ consists of those equivalence classes of vectors $(x_1, ..., x_{n+1})$ with $x_{n+1} \neq 0$. Any equivalence class like this has a unique representative of the form $(x_1, ..., x_n, 1)$, which we can identify with the point $(x_1, ..., x_n) \in \boldsymbol{R}^n$. What is left over is equivalence classes of vectors of the form $(x_1, ..., x_n, 0)$, and this is a copy of $\boldsymbol{RP}^{n-1}$.

**Cell Structure of Projective Space:** The relation above can be iterated, to give

$$\boldsymbol{RP}^n = \boldsymbol{R}^n \cup \boldsymbol{R}^{n-1} \cup ... \cup \boldsymbol{R}^0,$$

the last space being a single point. You can think of this in terms of coordinates: Continuing the analysis from the previous section, $\boldsymbol{R}^{n-1}$ is the set of equivalence classes of the form $(x_1, ..., x_{n-1}, 1, 0)$ and $\boldsymbol{R}^{n-2}$ is the set of equivalence classes of vectors of the form $(x_1, ..., x_{n-2}, 1, 0, 0)$, and so on.

Here is another way to think about this decomposition. We can build the circle $S^1$ by attaching two 1-balls to a pair of points. We think of $S^1$ as the equator of $S^2$ and we get $S^2$ by attaching two disks to the circle. We think of these two disks as the hemispheres. And so on. This procedure leads to a description of $S^n$ as a cell-complex with 2 cells in each dimension. The whole process is invariant with respect to the antipodal map, and so we get a cell decomposision of $\boldsymbol{RP}^n$ in which there is one cell in each dimension, up to and including $n$. This is another way to think about the equation above.

**Projective Transformation:** A *projective transformation* of $\boldsymbol{RP}^n$ is a map induced from an invertible linear transformation of $\boldsymbol{R}^{n+1}$. Such linear transformations are continuous and respect the equivalence relations. So, they induce continuous maps on $\boldsymbol{RP}^n$. The same applies to the inverse linear transformation, so these maps are homeomorphisms.

If you know about groups, you'll appreciate the statement that the set of projective transformations of $\boldsymbol{RP}^n$ forms a group. This group is denoted $PGL_n(\boldsymbol{R})$. It is an example of what is called a *Lie group*. In particular, it is a topological group on the sense discussed in §4.3 in the book. I'll discuss this later on in lecture.

A *line* in $\boldsymbol{RP}^n$ is the set of equivalence classes of vectors all contained in a single 2-plane through the origin. Put another way, a line is the image of a 2-plane through the origin under the quotient map. Given the definition of linear maps, projective transformations map lines to lines. Again, they are continuous homeomorphisms of $\boldsymbol{RP}^n$ which map lines to lines.

Consider a special case: $n = 2$. In this case, $\boldsymbol{RP}^2 = \boldsymbol{R}^2 \cup \boldsymbol{RP}^1$. Most lines of $\boldsymbol{RP}^2$ intersect $\boldsymbol{R}^2$ in an ordinary line, but there is one line, namely $\boldsymbol{RP}^1$, which is disjoint from $\boldsymbol{R}^2$. So, to figure out what projective transformations to do $\boldsymbol{RP}^2$ you can look a homeomorphisms which map lines to lines in the ordinary sense, except that sometimes points in the plane get mapped to points in $\boldsymbol{RP}^1$ "at infinity" and sometimes points "at infinity" get mapped into the plane.

The space of invertible linear transformations of $\boldsymbol{R}^3$ is a 9-dimensional manifold, because there are $9 = 3 \times 3$ entries of a matrix. Two matrices which are scalar multiples of each have the same action on $\boldsymbol{RP}^2$, and this makes $PGL_3(\boldsymbol{R})$ an 8-dimensional space. Call 4 points in $\boldsymbol{RP}^2$ a *quad* if no three lie in the same line.

The basic fact is that, given any two (ordered) quads, there is a unique projective transformation which maps the one to the other. Here is a sketch of the proof: Note that there is a unique linear transformation that maps any basis to any other. This means that there is always a projective transformation that maps any triple of points (not on a line) to the equivalence classes of the standard basis vectors. Then, fooling around with diagonal matrices, which all preserve these standard equivalence classes, you can arrange for the 4th point to do what you want, and in a unique way.

As a fun drawing exercise, let $ABCD$ be the unit square in $\boldsymbol{R}^2$ and sketch the action of the projective transformation that maps $A, B, C, D$ to $A, B, D, C$. In other words, two points are fixed and two are switched. This is the exercise we started in class.

**Complex Projective Space:** $\boldsymbol{CP}^n$ is defined to be the space $X/\sim$ where $X = \boldsymbol{C}^{n+1} - \{0\}$ and two vectors are equivalent if and only if they are *complex* multiples of each other.

Another way to understand this space is to let $A$ be the unit sphere in $\boldsymbol{C}^n$. Then $A$ is a full compact subset of $X$. After some effort, you can verify that $A/\sim$ is Hausdorff, and so the general principle says that $X/\sim$ is homeomorphic to $A/\sim$. In particular, $\boldsymbol{CP}^n$ is compact and Hausdorff. Two unit vectors in $A$ are equivalent if and only if they are unit complex multiples of each other. In other words, $\boldsymbol{CP}^n$ is obtained from the $2n$-dimensional sphere by crushing certain great circles to points.

As a special case, consider $\boldsymbol{CP}^1$. There is a map from $\boldsymbol{C}^2$ into $\boldsymbol{C} \cup \infty$ given by the map

$$f(z_1, z_2) = z_1/z_2.$$

This map respects the equivalence relations and induces a continuous bijection from the compact $\boldsymbol{CP}^1$ to $\boldsymbol{C} \cup \infty$, the 1-point compactification of $\boldsymbol{C}$, which we know to be homeomorphic to $S^2$. Since $\boldsymbol{CP}^1$ is compact and $S^2$ is Hausdorff, we see that $\boldsymbol{CP}^1$ is homeomorphic to $S^2$. This probably looks like the world's strangest description of the sphere.

In general, we have the same decomposition

$$\boldsymbol{CP}^n = \boldsymbol{C}^n \cup \boldsymbol{CP}^{n-1} = \boldsymbol{C}^n \cup \boldsymbol{C}^{n-1} \cup ... \cup \boldsymbol{C}^0.$$

So, $\boldsymbol{CP}^n$ is the union of Euclidean spaces of all even dimension up to and including $2n$. In particular, $\boldsymbol{CP}^2$ is homeomorphic to a union of $\boldsymbol{C}^2 \approx \boldsymbol{R}^4$

and $\boldsymbol{C} \approx \boldsymbol{R}^2$ and a point. This is pretty hard to visualize, but after some practice you can get used to working with it.

Complex projective space is extremely important an algebraic geometry.

**Stiefel Manifolds:** The space $V(k,n)$ is defined to be the space of all orthonormal $k$-frames in $\boldsymbol{R}^n$. (That is, all vectors are unit vectors, and every two are perpendicular.) We can make this space into a metric space by declaring that the distance between frames $(v_1,...,v_k)$ and $(w_1,...,w_k)$ is the maximum angle between some $v_i$ and some $w_i$. The symmetry condition is easy to check, and the triangle inequality is not too bad. This turns $V(k,n)$ into a metric space and hence a Hausdorff topological space.

Here's a rough sketch that $V(k,n)$ is a topological manifold of dimension $(n-1)+(n-2)+...+(n-k)$. For instance $V(2,4)$ has dimension $3+2=5$. Starting with a frame $(v_1,...,v_k)$ how do we move around? We first wiggle $v_1$ within a little open ball on the unit sphere $S^{n-1}$. Once we settle on a choice for $v_1'$, we can wiggle $v_2$ around within a little open ball in the copy of $S^{n-2}$ that is perpendicular to $v_1'$. Once $v_1'$ and $v_2'$ are chosen, we can wiggle $v_3$ around in a little open ball on the copy of $S^{n-3}$ that is perpendicular to both $v_1'$ and $v_2'$. And so on. In all cases, we are taking our choices from little open balls of dimension $n-1, n-2, ..., n-k$. This reveals the dimension to be as advertised, and a small neighborhood of the original frame to be a kind of product of open balls of various dimensions. Such a product is again homeomorphic to a Euclidean ball.

Since $V(k,n)$ is metric space, it is automatically Hausdorff. Here is an argument for compactness. We can identify $V(k,n)$ as a certain subset of $(\boldsymbol{R}^n)^k$ just by listing out the vectors. The subset we get is bounded because all vectors have unit length. It is also closed because any limit of a sequence of frames is again a frame. Informally, you would say that orthogonality is a closed condition. By the Heine-Borel theorem, this copy of $V(k,n)$ is compact.

All in all $V(k,n)$ is a compact Hausforff manifold whose dimension is $(n-1)+...+(n-k)$.

**Grassmann Manifolds:** The space $G(k,n)$ is defined as the space of $k$-planes through the origin in $\boldsymbol{R}^n$. There is a natural map from $V(k,n)$ to $G(k,n)$: You just map a $k$-frame to the $k$-plane that is spans. This map is surjective, so we topologize $G(k,n)$ with the quotient topology. There are ways to directly make $G(k,n)$ into a metric space, in terms of various "angles" between $k$-planes. I'll leave that to you. It is fun to think about.

The space $G(1,n)$ is just $\boldsymbol{RP}^{n-1}$. The spaces $G(k,n)$ and $G(n-k,n)$ have canonical homeomorphisms between then: Just map a $k$-plane to its perpendicular complement, which is an $(n-k)$ plane. So, the space $G(n-1,n)$ is also a copy of $\boldsymbol{RP}^{n-1}$. The first new space of $G(2,4)$.

There are various ways to see that $G(2,4)$ is homeomorphic to $S^2 \times S^2$, the product of 2 two-dimensional spheres, but I don't know a totally elementary way to do it. One way involves identifying $\boldsymbol{R}^4$ with the set of quaternions and then using properties of quaternionic multiplication. I won't get into this in these notes.

In general, the map from $V(k,n)$ to $G(k,n)$ is not injective because many orthonormal $k$-frames span the same $k$-plane. In fact, the space of orthonormal $k$-frames in the same $k$-plane has dimension $(k-1)+(k-2)+...$ This is a very rough sketch of the reason that $G(k,n)$ has dimension

$$(n-1)+...+(n-k)-(k-1)-(k-2)...$$

For instance $G(2,4)$ has dimension $3+2-1=4$. You might enjoy trying to figure out the dimension of $G(k,n)$ in your own way.