Math 1410: The Polygonal Jordan Curve Theorem: The purpose of these notes is to prove the polygonal Jordan Curve Theorem, which says that the complement of an embedded polygonal loop in the plane has exactly two connected components. (Since the complement is open, the components coincide with path components.)

Line Segments: Polygonal loops and paths are built from line segments, so let me say a few words about these first. Two line segments A and B *abut* if they share an endpoint and otherwise are disjoint. In other words, $A \cap B$ is an endpoint of both A and B. These segments are *transverse* if they are either disjoint or if they have a single intersection point which is in the interior of both A and B. If A and B abut, they are not transverse. If they are transverse, they do not abut. It is also possible that A and B are not transverse and do not abut.

Paths and Loops: A *polygonal path* is a finite union $A_1, ..., A_n$ of line segments such that A_i and A_{i+1} abut for all $i = 1, ..., n-1$ and otherwise A_i and A_j are transverse. The path is *embedded* if A_i and A_j are disjoint when $|j - i| > 1$. The polygonal path above is a *loop* if the first endpoint of A_1 coincides with the last endpoint of A_n. Such a loop can either be embedded or not. Figure 1 shows two polygonal loops, the first of which is embedded.

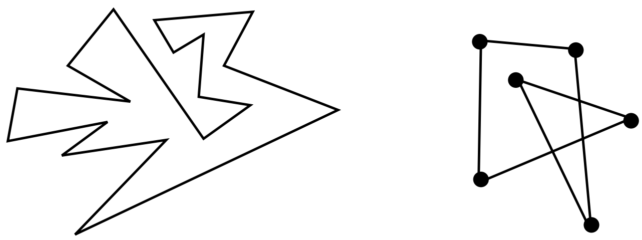

Figure 1: Two polygonal loops

When the loop or path is not embedded, it is useful to mark the vertices of the loop – i.e., the endpoints of the intervals – because from the picture alone it might not be entirely clear how we have decomposed the path/loop into segments. In the embedded case, there is nothing to worry about, unless two consecutive segments are parallel. We call a point of a polygonal loop *ordinary* if it lies on a unique line segment in the loop. Aside from the vertices, the polygonal loop at right in Figure 3 has three points which are not ordinary.

Transverse Intersections: Let A and B be two polygonal loops. We say that A and B intersect *transversely* if each segment comprising A is transverse to each segment comprising B, and if all the intersection points are ordinary. Figure 2 shows a transverse intersection between two polygonal loops.

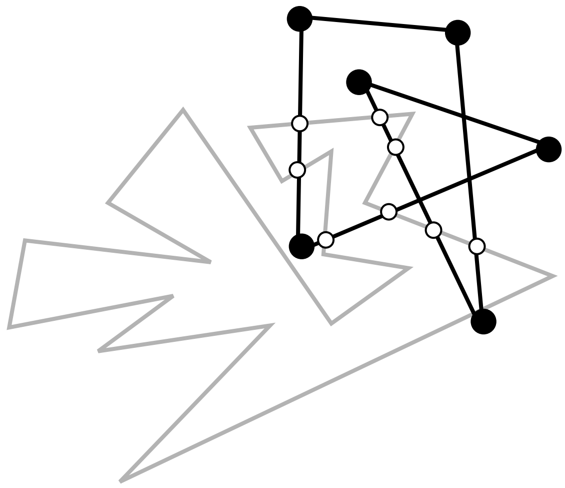

Figure 2: Two transversely intersecting loops.

Say that an *ϵ-perturbation* of A is a polygonal loop A', consisting of the same number of segments, such that the endpoints of each segment of A are within ϵ of the endpoints of the corresponding segment of A'. In other words, we just jiggle the vertices a bit to get A' from A. Here is a useful property of transverse intersection points.

Lemma 0.1 *Suppose A and B have a transverse intersection. If A' is an ϵ-perturbation of A and ϵ is sufficiently small then A' and B also have a transverse intersection and the number of intersection points of $A \cap B$ is the same as the number of intersection points of $A' \cap B$.*

Proof: The basic underlying principle we use is that the intersection point of two non-parallel segments varies continuously with the segments.

Let A_i and B_j be two segments of A and B respectively and let A_i' be the corresponding segment of A'. If A_i and B_j are disjoint then there is some minimum distance between them. If A_i' is sufficently close to A_i then A_i'

and B_j are also disjoint. On the other hand, if A_i and B_j intersect then the intersection point lies in the interior of both, and the two segments are not parallel. If ϵ is sufficiently small, A'_i and B_j also have a single intersection point that lies in the interior of both. This uses the continuity principle mentioned at the beginning of the proof.

There is some minimum distance between any point of $A \cap B$ and a non-ordinary point of A or B. This means that $A' \cap B$ will not contain any non-ordinary points as long as A' is sufficiently close to A. This uses the continuity principle once again. Now we know that A' and B have a transverse intersection.

Since A and B have a transverse intersection, the number of intersection points is just the total number of indices (i, j) where A_i intersects B_j. But the same goes for A' and B. Hence $A \cap B$ and $A' \cap B$ have the same number of intersection points. ♠

Remark: We stated the above definitions and results in terms of loops, but all the same definitions, results, and proofs work with paths in place of loops. I mention this because, below, I'll sometimes use the path case of the transversality result.

Lemma 0.2 *If A and B are polygonal loops with a transverse intersection, then $A \cap B$ has an even number of intersection points.*

Proof: First consider the case when A is a triangle. In this case, $\boldsymbol{R}^2 - A$ consists of two path components, the region bounded by the triangle and the region outside the triangle. We'll call these the *sides* of A. The number of times each segment of B intersects A is even if both endpoints of B are on the same side of A and odd if the endpoints lie in different sides. The transversality condition guarantees that the each vertex lies on one side or the other. Since the initial and final endpoint coincide, they lie on the same side of B. But this shows that there must be an even number of times when the endpoints of a segment of A lie on different sides of B. Hence the total number of intersection points is even.

Now consider the general case. The proof goes by induction on the number of sides of A. Let A^* be the polygonal loop obtained by replacing the first two edges of A with a single segment joining the endpoints of $A_1 \cup A_2$. Call this new segment A_{12}. Figure 3 shows A and A^*.

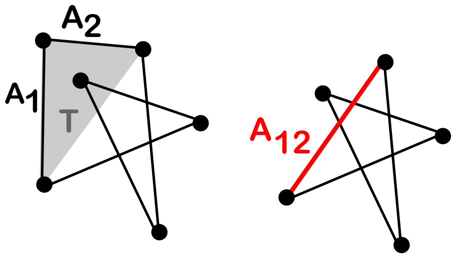

Figure 3: A and A^* and T.

Taking an ϵ-perturbation of A if necessary, we can arrange that A^* is also a polygonal loop, and both A and A^* have transverse intersection with B. (We perturb because we're worried about the stupid possibility that the side T_{12} is not transverse to some of the other edges of A.) Let T be the triangle whose sides are T_1, T_2, T_{12}. Letting $N(A, B)$ be the number of points of $A \cap B$, etc., we have

$$N(A, B) = N(A^*, B) + N(T, B) - 2N(A_{12}, B).$$

The sum $N(A^*, B)$ is even by induction, because A^* has one fewer side than A. The term $N(T, B)$ is even by the special triangle case. The third term is obviously even. So $N(A, B)$ is even because it is the sum of even numbers. ♠

The Sides of a Loop: Let A be a polygonal loop. We choose some point X which lies much farther from the origin than any point of A. For each point $p \in \boldsymbol{R}^2 - A$ we define $N(p)$ to be the parity, even or odd, of the number of times polygonal loop joining p to X intersects A. We always take our polygonal loops to be transverse to A, so that the intersection number is finite and the parity makes sense.

Lemma 0.3 *$N(p)$ is well-defined independent of polygonal loop chosen.*

Proof: Let B_1 and B_2 be two polygonal paths joining p to X. Since both B_1 and B_2 are transverse to A, the number of intersection points does not change if we replace B_1 and B_2 by small perturbations. We perturb, if necessary, so that $B = B_1 \cup B_2$ is a polygonal loop that has transverse intersection with A. Since B intersects A an even number of times, the number of times B_1 intersects A has the same parity as the number of times B_2 intersects A. ♠

Lemma 0.4 *Suppose that p_1 and p_2 are two points of $\boldsymbol{R}^2 - A$. Let B_0 be a polygonal path joining p_1 to p_2 which has transverse intersection with A. Then the parity of $A \cap B_0$ is the same as the parity of $N(p_1) + N(p_2)$.*

Proof: We denote the number of intersections by $N(*,*)$. We can join p_j to X by a polygonal path B_j. Perturbing our paths slightly, we can arrange that $B_0 \cup B_1 \cup B_2$ is a polygonal loop that intersects A transversely. By the parity result above,

$$N(A, B_0) + N(A, B_1) + N(A, B_2) \equiv N(A, B_0) + N(p_1) + N(p_2) \equiv 0 \text{ mod } 2.$$

This proves what we want. ♠

Corollary 0.5 *Suppose that $N(p_1) \neq N(p_2)$. Then p_1 and p_2 lie in different components of $\boldsymbol{R}^2 - A$.*

Proof: If p_1 and p_2 lie in the same component, then we can join them by a polygonal path that intersects A zero times. This contradicts the previous result. ♠

The above corollary proves half of the polygonal Jordan Curve Theorem. If A is any polygonal loop, embedded or not, then $\boldsymbol{R}^2 - A$ has at least two components. If remains to show, in the embedded, case that $\boldsymbol{R}^2 - A$ has exactly two components. We'll start with a technical construction and then give the main proof.

The Shadow Path: Suppose now that A is an embedded loop. We say that an ϵ-*shadow* is another polygonal path A' which which is disjoint from A and is less than ϵ from A in the Hausdorff metric. This means that each point of A is within ϵ of A' and vice versa.

Lemma 0.6 *Let $\epsilon > 0$ be given. There exists some $\delta > 0$ such that any point p that lies within δ of A lies on an ϵ-shadow of A.*

Proof: Let $A_1, ..., A_n$ be the sides of A. The shadow is going to have sides $A_1', ..., A_n'$, with A_i and A_i' parallel for all i. Let $L_1, ..., L_n$ be the lines through the vertices of A which bisect the angles at each of the vertices. This is shown in Figure 4.

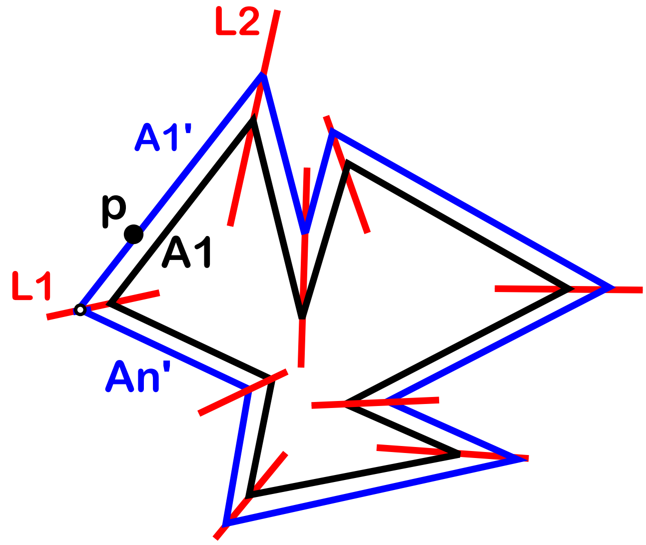

Figure 4: The loop in red and the shadow in blue.

Suppose without loss of generality that p is closest to A_1. Let A'_1 be the line segment which is parallel to A_1, contains p, and has endpoints on L_1 and L_2. Then let A'_2 be the line segment that has endpoints on L_2 and L_3, is parallel to A_2, and has a common endpoint with A'_1. Now define $A'_3, A'_4, \ldots$. The perpendicular distance between A_i and A'_i is the same as the perpendicular distance between A_{i+1} and A'_{i+1} by symmetry, because the reflection in the line L_{i+1} swaps the line containing A_i with the line containing A_{i+1} and also swaps the line containing A_{i+1} with the line containing A'_{i+1}. (Phil pointed this out in class.)

If the final endpoint of A'_n lies on the opposite side of L_1 from the endpoint of A'_1, then we can join these points togetether by the relevant segment of L_1, and this produces a polygonal loop that intersects A exactly once. This contradicts the parity result. Hence, the final endpoint of A'_n lies on the same side of L_1 as does the initial point of A'_1. Given our result about the perpendicular distances, we see that the final endpoint of A'_n coincides with the initial endpoint of A'_1. So, A' is a cloed loop.

A'_i is disjoint from A_i because these two segments are parallel. $A_{i'}$ is disjoint from A_{i+1} because these segments are separated by L_i. The same argument works for A_{i-1}. For the remaining segments, we just have to pick δ so small that A'_i is closer to A_i than the minimum distance between A_i and A_j for $|i-j|>1$. This guarantees that A'_i is disjoint from A. We do this for all indices i, and get A' disjoint from A. ♠

The End of the Proof: Suppose again that A is an embedded polygonal loop. Let $p_1, p_2 \in \boldsymbol{R}^2 - A$.

Lemma 0.7 *Suppose that $N(p_1) = N(p_2)$, and that p_1 and p_2 are joined by a polygonal path that is transverse to A, and $A \cap B$ has $N \geq 2$ points. Then p_1 and p_2 are also joined by a polygonal path that is transverse to A and intersects A less than N times.*

Proof: Figure 5 shows the idea of the proof. Let B be the original path joining p_1 to p_2. We follow along B until we come extremely close to the first intersection point of $A \cap B$. We then follow a detour along a shadow curve A' until we reach B again. (We must reach B again because $A \cap B$ has more than one point.) We then let B' be the path obtained by taking the detour. By construction, B' intersects A fewer times than B does. ♠

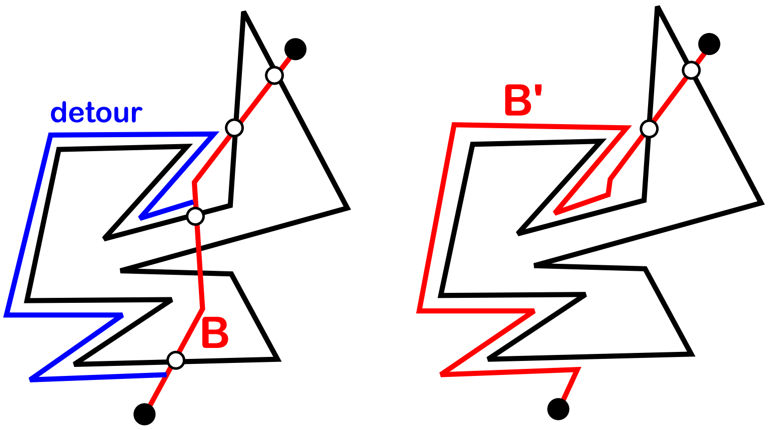

Figure 5: The origial path and the new path.

Corollary 0.8 *Let $p_1, p_2 \in \boldsymbol{R}^2 - A$. Suppose that $N(p_1) = N(p_2)$. Then p_1 and p_2 lie in the same component of $\boldsymbol{R}^2 - A$.*

Proof: Choose some polygonal path which joins p_1 to p_2 and is transverse to A and intersects A the minimum possible number of times. If this number, N, is positive, then $N \geq 2$. But then the previous lemma applies, showing that our path is not minimal. ♠

The two corollaries together say that $\boldsymbol{R}^2 - A$ has two components. One of them consists of the points p with $N(p) = 0$ and the other one consists of points p where $N(p) = 1$. Figure 6 shows an example, with the $N = 1$ component filled in.

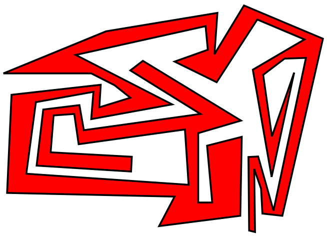

Figure 6: An embedded loop and one of its components.