Math 1410: Notes on the Reals: The purpose of these notes is to define the real numbers in terms of the rationals and then to prove some useful properties about them. The notes will assume all the basic properties of the rational numbers that you learn in your youth. Also, the notes have a do-it-yourself flavor. I'll give the basic constructions and definitions, and then you'll have some exercises to check the details. Doing the exercises is a really good way to learn this stuff well. You don't need to read these notes to follow the class, but these notes will enhance your understanding of all the arguments which explicitly mention $\boldsymbol{R}^n$.

Cauchy Sequences: An infinite sequence $\{a_i\}$ of rational numbers is called *Cauchy* if for every positive integer N there is another positive integer M such that

$$|a_i - a_j| < \frac{1}{N} \tag{1}$$

provided that both i and j are greater than M. You might say that a Cauchy sequence of rationals settles down.

Given two Cauchy sequences $\{a_i\}$ and $\{b_i\}$ we define their *shuffle* to be the sequence

$$a_1, b_1, a_2, b_2, a_3, b_3, \ldots \tag{2}$$

We say that $\{a_i\}$ and $\{b_i\}$ are *equivalent* if their shuffle is also a Cauchy sequence. In this case, we write $\{a_i\} \sim \{b_i\}$. Intuitively, this means that both sequences are "settling down in the same place".

Exercise 1: Prove that this notion of equivalence is an equivalence relation. In other words,

- Every Cauchy sequence is equivalent to itself.
- If $\{a_i\} \sim \{b_i\}$ then $\{b_i\} \sim \{a_i\}$.
- If $\{a_i\} \sim \{b_i\}$ and $\{b_i\} \sim \{c_i\}$ then $\{a_i\} \sim \{c_i\}$.

The *equivalence class* of a Cauchy sequence is the set of all Cauchy sequences equivalent to it. The set of all Cauchy sequences is partitioned into these equivalence classes. A *representative* of an equivalence class of Cauchy sequence is any Cauchy sequence belonging to that class.

Real Numbers: A real number is defined to be an equivalence class of

Cauchy sequences. Each real number is a different equivalence class. The set of all real numbers is denoted $\boldsymbol{R}$.

First of all, let's make sure that we haven't lost the rationals. For each rational number $a \in \boldsymbol{Q}$, we let a' denote the equivalence class of the Cauchy sequence $a, a, a, ...$ The map $a \to a'$ is an injective map from $\boldsymbol{Q}$ into $\boldsymbol{R}$. If we identify $\boldsymbol{Q}$ with the subset $\{a' |\ a \in \boldsymbol{Q}\}$ then we have a copy of $\boldsymbol{Q}$ sitting inside $\boldsymbol{R}$. In particular 0 is the equivalence class containing the Cauchy sequence $0, 0, 0, ...$ and 1 is the equivalence class of the sequence $1, 1, 1, ...$

The Field Operations: Suppose that α and β are two real numbers and $*$ is any of the symbols $+, -, \times, /$, we define $\alpha * \beta$ to be the equivalence class of the sequence $\{a_i * b_i\}$ where $\{a_i\}$ is a representative of α and $\{b_i\}$ is a representative of β. In case we are doing division, we only define this if $\beta \neq 0$.

Exercise 2: Prove that the operation $\alpha * \beta$ is well defined. In other words, if we take different representatives for α and β and do the same construction, we end with the same equivalence class of Cauchy sequences.

Exercise 3: Verify that the operations above turn $\boldsymbol{R}$ into a *field*. Those of you who have had an abstract algebra class know what this means, but for anyone else here are the properties you want to verify:

- $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \boldsymbol{R}$.
- $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ for all $\alpha, \beta, \gamma \in \boldsymbol{R}$.
- $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ for all $\alpha, \beta, \gamma \in \boldsymbol{R}$.
- $0 + \alpha = \alpha$ and $1 \times \alpha = \alpha$ for all $\alpha \in \boldsymbol{R}$.
- For all $\alpha \in \boldsymbol{R}$ there is a (unique) $\beta \in \boldsymbol{R}$ such that $\alpha + \beta = 0$.
- For all $\alpha \in \boldsymbol{R} - \{0\}$ there is a (unique) $\beta \in \boldsymbol{R}$ such that $\alpha\beta = 1$.

This is a fairly imposing list, but once you do a few of them you'll see that the verifications are all more or less the same, and follow from similar properties of the rationals.

Ordering the Reals: Suppose $\alpha \neq 0$ is a real number. We say that α

is *positive* if, for any representative $\{a_i\}$ of α, we have $a_i > 0$ once i is sufficiently large.

Exercise 4: Prove that this notion of positive is well-defined. That is, it is independent of the chosen representative. This result depends crucially on the fact that $\alpha \neq 0$.

Given two arbitrary reals α and β, we write $\alpha < \beta$ if and only if $\beta - \alpha$ is positive.

Exercise 5: Prove that this notion of order makes $\boldsymbol{R}$ into an *ordered Archimedian field*. That is, prove that

- If $\alpha < \beta$ and $\beta < \gamma$ then $\alpha < \gamma$.
- Given two unequal reals α and β, either $\alpha < \beta$ or $\beta < \alpha$.
- If $\alpha < \beta$ and $\gamma > 0$ then $\alpha\gamma < \beta\gamma$.
- If $\alpha > 0$ then there exists some positive integer N so that $N\alpha > 1$.

We define $\alpha \leq \beta$ if either $\alpha = \beta$ or $\alpha < \beta$. We define the other symbols $>$ and $\geq$ in the way you might expect.

The Least Upper Bound Property: A subset $S \subset \boldsymbol{R}$ is called *bounded from above* if there is some real number α such that $\sigma \leq \alpha$ for all $\sigma \in S$. The number α is called an *upper bound* for S.

The number α is called a *least upper bound* for S if there is no $\beta < \alpha$ such that β is also an upper bound for S. Now I'm going to explain why $\boldsymbol{R}$ has the least upper bound property: Each set S bounded from above has a least upper bound.

Let $2^{-n}\boldsymbol{Z}$ denote the set of rational numbers of the form $k/2^n$. This set is an infinite set of rational numbers which are evenly spaced. Note that $2^{-n}\boldsymbol{Z} \subset 2^{-(n+1)}\boldsymbol{Z}$. Let A_n denote the set of upper bounds for S in $2^{-n}\boldsymbol{Z}$. The set A_n is bounded from below. Therefore, A_n has a least element a_n. The number a_n is a rational number. By construction $a_{n+1} \leq a_n$ for all n. Hence $\{a_i\}$ is a monotone decreasing sequence which is bounded from below.

Exercise 6: Prove that a monotone decreasing sequence that is bounded from below is a Cauchy sequence.

Exercise 7: Let α be the equivalence class of the sequence $\{a_i\}$. Prove that α is the least upper bound for S. (Hint: After showing that α is an upper bound, try to reason like this: If $\beta < \alpha$ is another upper bound, show that some a_n is not the least element of A_n.

The least upper bound for S is denoted $\limsup S$. So, the real numbers have the property that every set bounded from above has a least upper bound. In summary, the real numbers are an Archimedian ordered field with the least upper bound property. The rest of the structures in $\boldsymbol{R}$ are defined in terms of the properties above. For instance (a, b) denotes the set of reals x with $a < x < b$. This is called an open interval. The open intervals form a basis for the usual topology on $\boldsymbol{R}$. Likewise $[a, b]$ is defined as the set of $x \in \boldsymbol{R}$ such that $a \leq x \leq b$. The set $[a, b]$ is called a *closed interval*.

At this point, you don't have to always keep in your mind that a real number is secretly an equivalence class of Cauchy sequences. You just have to remember that $\boldsymbol{R}$ is an Archimedian ordered field with the least upper bound property. There are several other constructions of $\boldsymbol{R}$ – e.g. one which uses something called *Dedekind cuts* – but all these constructions lead to a structure isomorphic to ours.

Nested Intervals and Cubes: Now we're going to prove a result that we'll use frequently in class. Let $\{I_n\}$ be an infinite sequence of closed intervals in $\boldsymbol{R}$. We call this sequence *nested* if $I_1 \supset I_2 \subset I_3, ...$ That is $I_{n+1} \subset I_n$ for all n.

Theorem 0.1 *Every nested sequence of intervals has at least one point in its intersection.*

Proof: The set S of left endpoints of these intervals is bounded from above. Let α be the least upper bound for this set. Each right endpoint is an upper bound for S. Therefore α is or equal to all the right endpoints. At the same time α is greater or equal to all the left endpoints. Therefore α belongs to every interval. ♠

Remark: Here is another way to think about the proof just given. The sequence of left endpoints is monotone increasing and bounded from above.

Therefore, it is a Cauchy sequence. The real number α constructed in the proof above is represented by this Cauchy sequence.

Exercise 8: Formulate and prove the same kind of result for nested sequences of cubes (or rectangular solids) in $\boldsymbol{R}^n$.