Notes on Sperner's Lemma

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1 Sperner's Lemma

Sperner's Lemma applies to any cleanly triangulated triangle. An example is shown on the left hand side of Figure 1. The drawing on the right is not meant to be an example; it shows something we want to avoid. The main things you want from the triangulation are that the small triangles meet edge to edge and pairwise have disjoint interiors. We call this a *clean triangulation*.

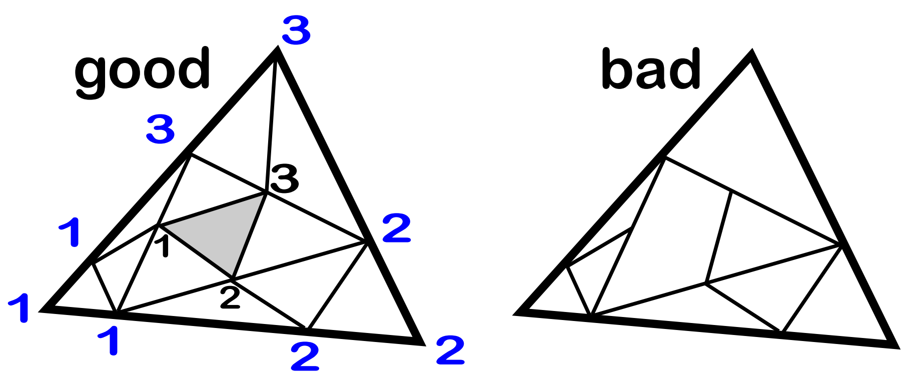

Figure 1: Two good triangulations and a bad one in action.

Let T be the triangle, and suppose it is cleanly triangulated into n smaller triangles $T_1, ..., T_n$. Suppose also that the vertices of the triangulation are labeled by integers $\{1, 2, 3\}$ so that the kth side of T has no k label. Sperner's Lemma says that some T_j gets all 3 labels. This result generalizes to n dimensions in a way which I will leave to you.

First Proof: Here is the reference for the amazing proof to follow.

A. McLennan and R. Tourkey, *Using volume to prove Sperner's lemma* Econ. Theory **35** (2008) pp 593-597

We're going to assume that we have a labeling in which no triangle sees all three labels and derive a contradiction. We normalize so that T has area 1. Let's call the vertices of the triangulation $V_1, ..., V_m$. Let $L_1, ..., L_m$ be the labels of these vertices. Let $W(1)$, $W(2)$, $W(3)$ be the vertices of the big triangle T. For each $t \in [0,1]$, define the new point

$$V_k(t) = (1-t)V_k + tW(L_k). \tag{1}$$

In other words, we think of t as time and we think of a the curve $t \to V_k(t)$ as a path which starts at V_k and ends at $W(L_k)$ and moves in a straight line at constant speed. Supposing that the triangle T_k has vertices A, B, C, let $T_k(t)$ be the triangle with vertices $A(t), B(t), C(t)$. As the points move with t these triangles change shape.

Consider the function

$$f(t) = \sum_{j=1}^{n} \text{area}(T_j(t)). \tag{2}$$

First of all, this function is a polynomial, thanks to the simple formulas – e.g. determinants – one can use to compute the areas of the triangles. Second of all, $f(t) = 1$ for all t sufficiently close to 0. The point is that, for small t, we still have a triangulation even though the points have moved a little. Since f is a polynomial, we must have $f(t) = 1$ for all $t \in [0,1]$. However, suppose that no triangle T_k sees all three labels. Then $T_k(t)$ converges either a single vertex or to an edge of T as $t \to 1$. Hence

$$\lim_{t \to 1} \text{area}((T_k(t)) = 0. \tag{3}$$

Since this is true for all triangles, and there are only finitely many of them in the triangulation, we see that $f(t) \to 0$ as $t \to 1$. This is a contradiction. That's the end of the proof.

Second Proof: This is the more traditional proof. Say that a *flag* is a pair (T_k, e), where T_k is one of the triangles of the triangulation and e is an edge of T_k. Each triangle participates in 3 flags and each edge either participates in 1 or 2 flags, depending on whether the edge is in the boundary of T. Say that (12)-flag is a flag which has the labels 1 and 2 on its edge. We're going to count the (12)-flags in two ways.

Let's first count these flags going edge to edge. Each interior edge contributes an even number of (12)-flags to the total, because it participates in two flags and these two flags are either simultaneously (12)-flags or not (12)-flags. The only boundary edges which contribute a (12)-flag are the ones on the side which has the 1 and 2 labels. This side is divided into finitely many edges. One endpoint of the side is labeled 1 and the other one is labeled 2. So, as we go from one endpoint to the other, we have to switch labels an odd number of times. Hence, there are an odd number of (12)-flags coming from the boundary edges. But that means there is an odd number of (12)-flags overall.

On the other hand, let's count the (12)-flags triangle by triangle. As I mentioned above, each triangle participates in 3 flags. Just list out the possibilities and you can see that a triangle contributes an odd number of (12)-flags to the count if and only if it gets all 3 labels. The triangles labeled $(1,2,2)$ $(1,1,2)$ each contribute two (12) flags to the count and the rest contribute zero. Hence, there must be an odd number of triangles which are labeled $(1,2,3)$. That completes the proof.

2 A Topological Interlude

Before going further, we need a foundational result from topology.

Lemma 2.1 (Uniform Continuity) *Suppose that X is a compact metric space and $f : X \to \boldsymbol{R}$ is a continuous function and $\epsilon > 0$ is given. Then there exists some $\delta > 0$ with the following property. For any two points $a, b \in X$ with $d(a, b) < \delta$¡ we have $|f(a) - f(b)| < \epsilon$.*

Proof: Suppose this is false. Then we can find sequences $\{a_n\}$ and $\{b_n\}$ such that $d(a_n, b_n) \to 0$ but $|f(a_n) - f(b_n)| \geq \epsilon$. By compactness, there is a subsequence of $\{a_n\}$ which converges to a point $a \in X$. Passing to a subsequence, we might as well assume that $a_n \to a$. But then $b_n \to a$ as well. Since f is continuous at a, there is some δ such that $|f(p) - f(a)| < \epsilon/2$ once $d(p, a) < \delta$. But than $|f(a_n) - f(b_n)| < \epsilon$ once n is large enough for both these points to be within δ of a. This contradiction proves the lemma. ♠

3 Applications

Suppose that X is a topological space and $Y \subset X$ is a subspace. A *retraction* from X to Y is a continuous map $f : X \to Y$ such that f is the identity map when restricted to Y. Here is a classic theorem from topology.

Theorem 3.1 *Let Δ be a topological disk. Let $\partial\Delta$ denote the boundary of Δ. Then there is no retraction from Δ to $\partial\Delta$.*

Proof: Since it is expressed entirely in terms of continuous maps, the notion of a retraction is invariant under homeomorphisms. So, we can assume that Δ is an equilateral triangle having side length 100.

We will suppose that f is a retraction and derive a contradiction. By the Uniform Continuity Lemma, we can find some δ so that $|f(a) - f(b)| < 1$ for all a, b in the triangle with $|a - b| < \delta$. Choose a triangulation of Δ where each triangle in the triangulation has side length less than δ.

Label a vertex v of the triangulation by the name of the vertex of Δ closest to $f(k)$. In case of a tie, choose the lower label. f maps the vertices of each small triangle so that they are each within 1 of each other. But then no triangle can be labeled $(1, 2, 3)$ because the three vertices of Δ are spread far apart. On the boundary, the labeling is more or less as in Figure 1. So, this labeling contradicts Sperner's Lemma. ♠

Theorem 3.2 *Suppose that Δ is a topological disk and $f : \Delta \to \Delta$ is a continuous map. Then f has a fixed point. That is, there is some $p \in \Delta$ such that $f(p) = p$.*

Proof: Again, this notion is invariant under homeomorphism, so we might as well assume that Δ is the standard round disk. We will suppose that f has no fixed points and derive a contradiction.

Given any $p \in \Delta$. consider the ray R_p which starts at $f(p)$ and moves in the direction of p. Let $g(p) = R_p \cap \partial\Delta$, the point where the ray intersects the boundary. It is easy to check that the function g is continuous. If you move

p a little, then the ray does not change much, either in terms of slope or starting point. So, it hits $\partial\Delta$ at a nearby point. At the same time, $g(p) = p$ when $p \in \partial\Delta$. Hence g is a retraction. Contradiction. ♠

Here is a nice application from linear algebra. Say that a *positive matrix* is a square matrix with all positive entries. Say that a *positive eigenvector* is an eigenvector with all positive entries. All the results above work in n dimensions, and have the same proofs. I'll leave this to you, but will assume the n-dimensional Brouwer fixed point theorem for the next result.

Theorem 3.3 *Any positive matrix has a positive eigenvector.*

Proof: Let Δ denote the space of rays which emanate from the origin and point into the positive orthant. One can put a metric on Δ as follows: The distance between two rays is the angle they make. With this metric, Δ is isometric to a spherical simplex. In particular, Δ is homeomorphic to a ball. The matrix M induces a continuous map from Δ into itself. By the Brouwer fixed point Theorme, M has an eigenvector E with non-negative coefficients. Note that $M(V)$ is a positive vector whenever V is a non-negative vector. Hence E must be a positive vector. ♠