Math 1410: Hairy Sphere Theorem:

The purpose of these notes is to prove the famous Hairy Sphere Theorem. Let S^2 be the unit sphere in $\boldsymbol{R}^3$. A *continuous vector field* on S^2 is a continuous choice of vector V_p for each point $p \in S^2$. The vector field is described by a triple of functions, so the continuity of the vector field just boils down to the continuity of these functions. For visualization purposes, one considers V_p to be based at p. The vector field is called *tangent* if V_p is tangent to S^2 at p, for all $p \in S^2$.

Theorem 0.1 (Hairy Sphere) *There is no non-zero continuous tangent vector field on S^2.*

Proof: We'll suppose that such a vector field V exists and derive a contradiction. The proof makes crucial use of the fact that $\pi_1(S^1, 1)$ is isomorphic to $\boldsymbol{Z}$. Here S^1 is the circle of unit complex numbers.

Let $\gamma : [0,1] \to S^2$ be a regular continuously differentiable loop. (This means that the derivative γ' is well defined at each point, and nonzero, and continuous.) Given γ we associate an auxiliary map $g : [0,1] \to S^1$ such that $g(0) = g(1) = 1$. We first define $\widehat{g}(t) = \exp(i\theta)$, where θ is the counterclockwise angle through which you have to rotate $\gamma'(t)$ until it is parallel to $V_{\gamma(t)}$. Next, we define $g(t) = \widehat{g}(t)/\widehat{g}(0)$. The map g has all the desired properties. The map g defines an element in $\pi_1(S^1, 1)$. So, we interpret $[\gamma]$, the element of $\pi_1(S^1, 1)$ as an integer.

To avoid having to deal explicitly with the parametrization, we always take γ so that it is a constant speed parametrization of a round circle. Once we specify the circle, a distinguished point on the circle, and a direction to go around, the map γ is uniquely determined. Both γ and the derivative γ' vary continuously with the data.

We specify a small circle C_0 centered at the north pole, some distinguished point p_0, and the clockwise direction. Let γ_0 be the corresponding map. In a neighborhood of the north pole, the vectors in the vector field all point nearly in the same direction. On the other hand, the derivative vectors along C_0 rotate once around clockwise. So, relative to these derivative vectors, the vector field rotates once around counterclockwise. Hence $[\gamma] = 1$. If we used the same data but specified the counterclockwise direction, we would get -1 instead.

Now for the punchline. We can steadily enlarge C_0, pulling it all the way over the sphere (as a family of lattitude circles) until is is a small circle around the south pole, then we can slide this small circle back up along a single line of longitude, keeping it the same size, until we return to C_0. We can keep the distinguished point on a fixed line of longitude so that it varies continuously. However, the direction reverses when we get back. So, we have found a continuous motion from $(C_0, p_0, +)$ to $(C_0, p_0, -)$ where $+$ and $-$ respectively denote the clockwise and counterclockwise direction choice.

Let γ_t be the map associated to the data at time t. We have $[\gamma_0] = 1$ and $[\gamma_1] = -1$. However, the continuous motion gives a homotopy between γ_0 and γ_1 which keeps the endpoints fixed. This proves that $1 = -1$, which is a contradiction. ♠

Corollary 0.2 *S^2 is not a topological group.*

Proof: We'll suppose that S^2 is a topological group and we'll derive a contradiction. We claim that there is some $h \in S^2$, not the identity, such that g and hg lie in the same hemisphere for all $g \in S^2$. Suppose this is true for the moment. Note that $g \neq hg$ because otherwise we would have

$$e = gg^{-1} = hgg^{-1} = h.$$

Since g and hg always lie in the same hemisphere, there is a uniquely defined and continuously varying shortest arc of a great circle joining g to hg. We define V_g to be the vector tangent to this arc. This constructs a continuous unit tangent vector field on S^2 contradicting the Hairy Sphere Theorem.

To finish the proof we just have to prove the claim. If this claim is false then, for any positive integer n, there are points $g_n, h_n \in S^2$ such that h_n is less than $1/n$ from e and the two points g_n and $h_n g_n$ do not lie in the same semicircle. By the Bolzano-Weierstrass Theorem, the set $\{g_n\}$ has a limit point g. Passing to a subsequence, we can assume that g_n converges to g.

Consider the map $F(a, b) = (a, ab)$. Since the group multiplication law is continuous, F is continuous as a map from $S^2 \times S^2$ to $S^2 \times S^2$. Note that $F(g, e) = (g, g)$, the same point repeated twice. Since F is continuous, there is some N such that g' and $h'g'$ lie in the same hemisphere provided that h' is within $1/N$ of e and g' is within $1/N$ of g. For n sufficiently large, the two elements $g' = g_n$ and $h' = h_n$ have this property. This gives us a contradcition. ♠