Math 153 Midterm 2 Solutions: Prof. Schwartz

1. First let's show that J is closed under addition. Suppose $a, b \in J$. We have

$$(a+b)^2 = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + b^2 \in I.$$

The last equality comes from the fact that R has characteristic 2. Hence $a + b \in J$. Since $(-a)^2 = a^2$, we see that J is also closed under taking additive inverses. To finish the proof, we have to show that J has the absorbtion property. If $r \in R$ and $a \in J$ then $(ra)^2 = r^2a^2 \in I$, because I is an ideal. This proves that $ra \in J$.

2. Let U be the union of 0 and the units in R. The set U has $N + 1$ elements. Let $\phi : R \to R/Rx$ be the quotient map. Here x is an element having smallest d-value amongst all non-units. Note that $\phi(U)$ has at most $N + 1$ elements. If $s \in R$ is any element then by the Euclidean domain property, we have $s = qx + r$ where either $r = 0$ or $d(r) < d(x)$. But, in either case, $r \in U$ and moreover $\phi(s) = \phi(r)$. Hence $\phi(R) = \phi(U)$. Since $\phi(U)$ has at most $N + 1$ elements, so does $\phi(R)$.

3: Suppose $\phi : 5\boldsymbol{Z} \to 7\boldsymbol{Z}$ is an isomorphism. Then $\phi(5) = 7k$ for some $k \in \boldsymbol{Z}$. But then

$$49k^2 = \phi(5)\phi(5) = \phi(25) = \phi(5 + 5 + 5 + 5 + 5) = 5\phi(5) = 35k.$$

Cancelling gives $7k = 5$, which is impossible for $k \in \boldsymbol{Z}$. This contradiction shows that no such isomorphism exists.

4. Let $d_0, ..., d_k$ be the digits of n. We have

$$n = \sum_{i=0}^{k} d_i 10^i, \qquad s = \sum_{i=0}^{k} d_i.$$

Let $\phi : \boldsymbol{Z} \to \boldsymbol{Z}/9\boldsymbol{Z}$ be the quotient homomorphism. We have

$$\phi(n) = \sum_{i=0}^{k} \phi(d_i)\phi(10^i) = \sum_{i=0}^{k} \phi(d_i)\phi(10)^i = \sum_{i=0}^{k} \phi(d_i)\phi(1)^i = \phi(s).$$

The various equalities are true because $1 \equiv 10 \bmod 9$ and ϕ is a ring homomorphism. Since $\phi(n) = \phi(s)$, the two numbers are congruent mod 9.

5. Consider the set $I' = \{a(x,y)x^2 + b(x,y)y^2\}$. It is easy to check that I' has the absorbtion property, and that I' is closed under addition and additive inverses. Hence I' is an ideal. I' contains both x^2 and y^2. Hence $I \subset I'$. At the same time, any ideal containing both x^2 and y^2 also contains $a(x,y)x^2 + b(x,y)y^2$ for any polynomials $a(x,y)$ and $b(x,y)$. Hence $I' \subset I$. In short $I = I'$.

Note that no nonzero constant polynomial belongs to I because neither $a(x,y)x^2$ nor $b(x,y)y^2$ has a constant term, and neither does their sum. This gives part (a).

Suppose $I = Rp$ for some polynomial $p(x,y)$. Then $x^2 = a(x,y)p(x,y)$. This is only possible if neither $a(x,y)$ nor $p(x,y)$ has any terms involving y. Hence $p(x,y)$ is a polynomial in x. But the same argument, using y^2 in place of x^2, shows that $p(x,y)$ is a polynomial in x. But then $p(x,y)$ is a constant polynomial. This contradicts the fact that I contains no nonzero constant polynomials.

6. Suppose that $\boldsymbol{Z}_p$ has zero divisors $\{a_n\}$ and $\{b_n\}$. Let A_n and B_n be integers in $\{0,...,p^n-1\}$ which represent a_n and b_n respectively in $\boldsymbol{Z}/p^n$. If $A_n = 0$ then $A_k = 0$ for all $k < n$. So, under the assumption that $\{a_n\}$ and $\{b_n\}$ are nonzero elements of $\boldsymbol{Z}_p$, it is not possible that A_n is zero infinitely often and it is not possible that B_n is zero infinitely often.

Let n be an arbitrary positive number. Since $a_{2n}b_{2n} = 0$ in $\boldsymbol{Z}/p^{2n}$, we must have $p^{2n}|A_{2n}B_{2n}$. But then either $p^n|A_{2n}$ or $p^n|B_{2n}$. Consider the former case. Since $A_{2n} \equiv A_n \mod p^n$, we see that $p^n|A_n$. But this forces $A_n = 0$. In the latter case, we get $B_n = 0$. Letting n vary, we see that A_n is zero infinitely often or B_n is zero infinitely often. Either case is a contradiction.