Math 153: Some Notes

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1 A Homework Problem

Here is a solution to one of the problems from the last HW assignment. Let R be a ring and suppose that the only right ideals of R are $\{0\}$ and R. We're going to show that R is either a division ring or $R = \boldsymbol{Z}/p$ with trivial multiplication. Trivial multiplication means that $ab = 0$ for all $a, b \in \boldsymbol{R}$.

Suppose first that R has trivial multiplication. Then R is commutative, and a subset of R is an ideal if and only if it is a right ideal if and only if it is a subgroup. We know from a previous problem that a group has no nontrivial subgroups if and only if it is isomorphic to $\boldsymbol{Z}/p$ for some prime p. So, R is isomorphic as a ring to $\boldsymbol{Z}/p$ with trivial multiplication. So, now we'll show that R is a division ring if R does not have trivial multiplication.

Say that a *bad element* of R is an element a such that $ab = 0$ for all $b \in \boldsymbol{R}$.

Lemma 1.1 *R has no bad elements.*

Proof: Let I be the set of bad elements. You can check that I is a right ideal. So, if there are any bad elements at all, I is a nontrivial ideal and hence $I = R$. But then R has trivial multiplication. Since this is not the case, there are no bad elements. ♠

Lemma 1.2 *For any nonzero $a \in R$ we have $aR = R$.*

Proof: aR is a nontrivial right ideal. Hence $aR = R$. ♠

Lemma 1.3 *R has no zero divisors.*

Proof: Suppose that $ab = 0$ for some nonzero a and some nonzero b. Since b is not a bad element, $bR = R$. But $abr = a(br)$ for all $r \in R$. Since $bR = R$, we see that $as = 0$ for all $s \in \boldsymbol{R}$. But then a is a bad element. This is a contradiction. ♠

Lemma 1.4 *R has an element λ such that $\lambda s = s$ for all $s \in \boldsymbol{R}$.*

Proof: Choose $a \in R$ nonzero. We have $aR = R$. Hence $a\lambda = a$ for some λ. But then $a\lambda\lambda = a\lambda = a$. So $a\lambda^2 = a\lambda$. But then $a(\lambda^2 - \lambda) = 0$. Since there are no zero divisors, we have $\lambda^2 = \lambda$. But then $\lambda(\lambda r) = \lambda^2 r = \lambda r$ for all $r \in R$. But $\lambda r = r$. Since λ is nonzero, $\lambda R = R$. Hence $\lambda s = s$ for all $s \in R$. ♠

Lemma 1.5 $\lambda = 1$.

Proof: For any nonzero a, we have $a\lambda a = aa$. Using cancellation, we get $a\lambda = a$. So, $\lambda a = a = a\lambda$ for all nonzero a. This shows that $\lambda = 1$. ♠

Now we can finish the proof. For any nonzero $a \in R$ we have $aR = R$. This means that $1 = ab$ for some $b \in \boldsymbol{R}$. At the same time $aba = a = a1$. Hence $ba = 1$ as well. This shows that every nonzero $a \in R$ has a multiplicative inverse. Hence R is a division ring.

Here is a nice corollary.

Corollary 1.6 *Suppose that R is a commutative ring and M is a maximal ideal and R/M is infinite. Then R/M is a field.*

Proof: R/M satisfies the hypotheses of the homework problem because ideals coincide with right ideals in the commutative case, and the ideals of R/M are in one-to-one correspondence with the ideals of R containing M. Since R/M is not $\boldsymbol{Z}/p$, it must be a division ring and therefore a field. ♠

2 More Maximal Ideals

In this section I'll prove the following result. If R is a commutative ring with 1, and $M \subset R$ is a maximal ideal (not equal to R) then R/M is a field.

2.1 First Proof

Since $M \neq R$, we have $1 \neq M$. But then the coset $[1]$ in R/M has the property that $[1][1] = [1]$. Hence R/M does not have trivial multiplication. The homework problem again applies, and shows that R/M is a division ring and hence a field.

2.2 Second Proof

We can reduce this to the result in the book if we can show that R/M is an integral domain. So, suppose for the sake of contradiction that R/M has zero divisors $[a]$ and $[b]$. We have $ab \in M$ but neither a nor b lies in M. Consider the set

$$I = \{ca + m |\ c \in R, m \in M\}.$$

This set is an ideal, and $M \subset I$. Since M has a 1, we have $a \in I$. But $a \notin M$. Hence $M \neq I$. Since M is maximal, $I = R$. But then $1 = ca + m$ for some $c \in R$. Multiplying through by b, we get $b = c(ab) = bm \in M$. This is a contradiction.