

The purpose of these notes is to explain the complex analysis you need to know for the Weierstrass uniformization of elliptic curves over \mathbf{C} .

1 A Resume of Results

Let U be an open set in \mathbf{C} , the complex plane. Let $f : U \rightarrow \mathbf{C}$ be a continuous map. We say that f has a complex derivative at $z \in U$ if the quotient

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite. Note that h is allowed to be a complex number. f is said to be *complex analytic* (or CA for short) in U if $f'(z)$ exists for all $z \in U$ and the function $z \rightarrow f'(z)$ varies continuously in U . Here are the basic results we're going to prove.

1. **Bounded Implies Constant:** Suppose $f : \mathbf{C} \rightarrow \mathbf{C}$ is CA and bounded, then f is constant.
2. **Removable Singularities:** Let U be an open set and let $b \in U$ be a point. Suppose that $f : U - \{b\} \rightarrow \mathbf{C}$ is CA and f is bounded in a neighborhood of b . Then f extends to all of U and is CA on U .
3. **Non-Vanishing:** Suppose that f is CA in a neighborhood of $a \in \mathbf{C}$, and not identically 0. Then there is some m such that $f(z+a) = z^m g(z)$ where g is CA in a neighborhood of 0 and $g(0) \neq 0$.
4. **Local Homeomorphism:** Let $f : U \rightarrow \mathbf{C}$ be a CA map. If $f'(a) \neq 0$ then there is some open disk Δ about a such that $f : \Delta \rightarrow f(\Delta)$ is a homeomorphism.

2 Cauchy–Riemann Equations

We can think of a CA function f as a map from \mathbf{R}^2 to \mathbf{R}^2 by writing

$$f(x + iy) = u(x + iy) + iv(x + iy).$$

To say that the complex derivative of f exists is the same as saying that f is differentiable and $df|_p$ (the differential at p) is the composition of a rotation

and a dilation. That is

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} r \cos(\theta) & r \sin(\theta) \\ -r \sin(\theta) & r \cos(\theta) \end{bmatrix}, \quad r \in \mathbf{R}, \quad \theta \in [0, 2\pi).$$

Equating terms, we get

$$u_x = v_y; \quad u_y = -v_x. \tag{1}$$

These are called the *Cauchy–Riemann equations*.

3 Complex Line Integrals

Suppose γ is a smooth oriented arc in \mathbf{C} and f is a complex valued function defined in a neighborhood of γ . We define a complex line integral along γ as follows. Letting $g : [a, b] \rightarrow \gamma$ be a smooth parametrization of γ that respects the orientation of γ , we define

$$\int_{\gamma} f \, dz = \int_a^b f(g(t)) \frac{dg}{dt} \, dt. \tag{2}$$

From the change of variables formula for integration, the answer only depends on the orientation and not the parametrization. Were we to switch the orientation, the value of the line integral would switch signs.

If we have a finite union $\gamma = \{\gamma_j\}$ of smooth oriented arcs, we define

$$\int_{\gamma} f \, dz = \sum_j \int_{\gamma_j} f.$$

Theorem 3.1 (Cauchy) *Let γ be a loop made from finitely many smooth arcs. Suppose that f is CA in a neighborhood of the domain bounded by γ . Then $\int_{\gamma} f \, dz = 0$.*

Proof: Let $f = u + iv$. Letting dx and dy be the usual line elements, we can write

$$\int_{\partial D} f \, dz = \int_{\partial D} (u + iv)(dx + idy) = \int_{\partial D} (udx - vdy) + i \int_{\partial D} (vdx + udy).$$

By Green's theorem, the integral on the right-hand side equals

$$\int_D (u_y + v_x) \, dx \, dy + i \int_D (u_x - v_y) \, dx \, dy.$$

Both pieces vanish, due to the Cauchy–Riemann equations. ♠

4 The Cauchy Integral Formula

Here is the main technical tool we will use.

Theorem 4.1 (Cauchy Integral Formula) *Let γ be loop oriented counterclockwise around the domain D that it bounds. Let $a \in D - \gamma$. Suppose that f is CA in a neighborhood U of D . Then*

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} dz. \quad (3)$$

Proof: We translate the whole picture and consider without loss of generality the case when $a = 0$. The function $g(z) = f(z)/z$ is CA in $U - \{0\}$. Let β be the circular polygon shown in Figure 1.

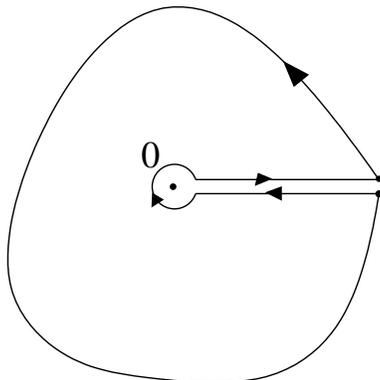


Figure 1.

We have

$$\int_{\beta} g dz = 0 \quad (4)$$

by Cauchy's Theorem. We allow the two oppositely oriented vertical segments in β to approach each other. In the limit, the contributions from the two vertical segments cancel out, and equation (4) yields

$$\int_{\gamma} g(z) dz = \int_{\lambda} g(z) dz. \quad (5)$$

Here λ is a counterclockwise circle centered at 0. But

$$\int_{\lambda} g(z) dz \approx f(0) \int_{\lambda} \frac{dz}{z} = 2\pi i f(0). \quad (6)$$

The approximation becomes exact as the radius of λ shrinks to 0. ♠

5 Bounded Functions

Suppose that $f : \mathbf{C} \rightarrow \mathbf{C}$ is a CA bounded function. We will show that $f'(a) = 0$ for all $a \in \mathbf{C}$.

Using the Cauchy Integral Formula, we compute

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \rightarrow 0} \frac{1}{2\pi i h} \left(\int_{\gamma} \frac{f(z)}{z - a - h} dz - \int_{\gamma} \frac{f(z)}{z - a} dz \right) = \\ &= \lim_{h \rightarrow 0} \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)(z - a - h)} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^2} dz. \end{aligned} \quad (7)$$

Equation 7 tells us that

$$f'(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^2} dz. \quad (8)$$

Let N be the radius of γ . The length of γ is $2\pi N$. The numerator in Equation 8 is at most C . The denominator is at least $|N - a|^2$. Hence,

$$|f'(a)| \leq \frac{CN}{|N - a|^2}. \quad (9)$$

Letting N tend to ∞ , we see that $|f'(a)| = 0$. But a is arbitrary. Hence $f'(a) = 0$ for all $a \in \mathbf{C}$.

6 Removable Singularities

Here we will prove the following result:

Theorem 6.1 *Let U be an open set that contains a point b . Suppose that f is CA and bounded on $U - \{b\}$. Then $f(b)$ can be (uniquely) defined so that f is CA in U .*

Proof: Let γ and β and λ be the loops used to prove the Cauchy Integral Formula. So, λ is a small loop surrounding b and γ is a big loop surrounding b . Let $|\lambda|$ denote the radius of λ . Let D be the open domain bounded by γ . We define $g : D \rightarrow \mathbf{C}$ by the integral

$$g(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} dz.$$

The same derivation used for Equation 7 shows that g is CA on all of D . We will show that $f(a) = g(a)$ for all $a \in D - \{b\}$. Once we know this, we set $f(b) = g(b)$ and we are done.

Now suppose that $a \neq b$. Since $f(z)$ is bounded in a neighborhood of b we have

$$\lim_{|\lambda| \rightarrow 0} \int_{\lambda} \frac{f(z)}{z - a} dz = 0.$$

But, by the Cauchy Integral Formula,

$$f(a) = \frac{1}{2\pi i} \int_{\beta} \frac{f(z)}{z - a} dz$$

no matter which choice of λ we make. Therefore

$$f(a) = \lim_{|\lambda| \rightarrow 0} \frac{1}{2\pi i} \int_{\beta} \frac{f(z)}{z - a} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} dz = g(a).$$

So $f(a) = g(a)$ for all $a \in D - \{b\}$. ♠

7 The Maximum Principle

Let f be a complex analytic function in a connected open set U . Here we will show that f cannot take on its maximum value at a point in U unless f is constant. We will assume that f takes on a maximum at some point $a \in U$, and we will derive a contradiction. If f has an interior maximum, we can compose f with translations and dilations and arrange the following properties.

- $|f(0)| = 1$.
- U contains the unit disk.
- $|F(z)| \leq 1$ for all $|z| = 1$.
- $|F(z)| < 1$ for some z such that $|z| = 1$.

Let γ be the unit circle. By the Cauchy Integral Formula we have

$$1 = |f(0)| = \frac{1}{2\pi} \left| \int_{\gamma} \frac{f(z)}{z} dz \right| \leq^* \frac{1}{2\pi} \int_{\gamma} |f(z)| dz < 1.$$

This is a contradiction. The starred inequality is essentially the triangle inequality.

8 Non-Vanishing

Lemma 8.1 *Suppose, for all n , that the function $f(z)/z^n$ is well defined at 0 and complex analytic in a neighborhood of the unit disk. Then f is identically 0 on the unit disk.*

Proof: Let $g_n(z) = f(z)/z^n$. Let M be the maximum of f on the unit disk. Note that $|g_n(z)| = |f(z)|$ for $|z| = 1$. But the maximum principle, $|g_n(z)| \leq M$ for all z in the unit disk. Hence $|f(z)| \leq M|z|^n$. If $|z| < 1$, then

$$\lim_{n \rightarrow \infty} M|z|^n = 0.$$

Hence $|f(z)| = 0$ if $|z| < 1$. By continuity, $|f(z)| = 0$ if $|z| \leq 1$. ♠

Now we prove Item 3. We want to see that $f(a+z) = z^n g(a)$ for some CA function g such that $g(a) \neq 0$. We can translate the picture so that $a = 0$. Then we can dilate the picture so that f is defined and CA in a neighborhood of the closed unit disk Δ . If $f(0) \neq 0$, then we are done. If $f(0) = 0$, then the existence of $f'(0)$ guarantees that the function $f(z)/z$ is bounded on $\Delta - \{0\}$. But then $f(z)/z$ is CA in Δ . Hence $f(z) = z g_1(z)$, where $g_1(z)$ is CA in Δ .

If $g_1(a) \neq 0$ we are done. Otherwise, by the same argument, we can write $f(z) = z^2 g_2(z)$. And so on. The only way Item 3 could fail is that there is a sequence of CA functions $\{g_n\}$ such that $f(z) = z^n g_n(z)$ for all n . But then f satisfies the hypotheses of Lemma 8.1. Hence f vanishes identically in the unit disk. This is a contradiction.

9 Local Homeomorphism

Now we prove Item 4. Let $f : U \rightarrow \mathbf{C}$ be a CA function. We call f *normalized* if f is defined at 0 and $f(0) = 0$ and $f'(0) = 1$.

Lemma 9.1 *When f is normalized, we have a small disk Δ centered at 0 and an estimate*

$$\left| (f(a) - f(b)) - (a - b) \right| \leq \frac{|a - b|}{10}, \quad (10)$$

which holds for all $a, b \in \Delta$.

Proof: f' exists and is continuous. Hence, there is some small neighborhood Δ about 0 with the property that

$$|f'(z) - 1| < 10^{-100} \tag{11}$$

for all $z \in \Delta$. If γ is any curve in Δ , then the velocity of γ at z and the velocity of $f(\gamma)$ at $f(z)$ are nearly the same, by Equation 11. To get our result, we compare the straight unit speed line segment γ connecting a to b with the nearly straight, nearly unit speed, curve $f(\gamma)$ connecting $f(a)$ to $f(b)$. ♠

In what follows, we always take Δ to be as in Lemma 9.1 when we are talking about normalized functions.

Lemma 9.2 *Suppose f is normalized. Then f is one to one on Δ and f^{-1} is continuous on $f(\Delta)$.*

Proof: The fact that f is one to one follows immediately from Equation 10. Equation 10 also gives us the bound $|f(a) - f(b)| > |a - b|/2$ for all $a, b \in \Delta$. But then we have $|f^{-1}(a) - f^{-1}(b)| < 2|a - b|$ for all $a, b \in f(\Delta)$. This bound obviously does the job. ♠

The next result has many different kinds of proofs, but none of the proofs is really easy. We'll give a geometrical proof. As you read this proof, you should think about a bad game of golf: The player hits the ball towards the hole but misses slightly. The next put comes closer, etc. In the limit, the ball goes in the hole.

Lemma 9.3 *Suppose f is normalized. Then $f(\Delta)$ contains a neighborhood of 0.*

Proof: Let r be the radius of Δ . Suppose $\alpha \in \mathbf{C}$ satisfies $|\alpha| < r/10$. Let $a_0 = \alpha$. We have

$$|a_0| < r/10; \quad |\alpha - f(a_0)| < r/100.$$

The second bound comes from Equation 10. In general, suppose that

$$|a_n| < r/10 + r/100 + \dots + r/10^n < r; \quad |f(a_n) - \alpha| < r/10^{n+1},$$

we define

$$a_{n+1} = a_n + \alpha - f(a_n).$$

Note that

$$|a_{n+1}| < r/10 + r/100 + \dots + r/10^{n+1} < r.$$

Hence $a_n, a_{n+1} \in \Delta$. Equation 10 then gives

$$\begin{aligned} |f(a_{n+1}) - \alpha| &= |(f(a_{n+1}) - f(a_n)) - (a_{n+1} - a_n)| \leq \\ &= \frac{|a_{n+1} - a_n|}{10} = \frac{|f(a_n) - \alpha|}{10} < r/10^{n+2}. \end{aligned}$$

So, we can define a_1, a_2, \dots inductively. By construction $|\alpha - f(a_n)| \rightarrow 0$ as $n \rightarrow \infty$. By continuity $f(a) = \alpha$ where $a = \lim a_n$. ♠

Lemma 9.4 *Suppose that $f : U \rightarrow \mathbf{C}$ is CA and $f'(a) \neq 0$ for some $a \in U$. Then*

- f is one to one in a neighborhood Δ of a .
- $f(\Delta)$ contains an open neighborhood of $f(a)$.
- $f^{-1} : f(\Delta) \rightarrow \Delta$ is continuous.

Proof: We can scale f so that $a = 0$ and f is normalized. This scaling does not change the conclusions of the lemma. This result now follows from everything we have proved about normalized functions. ♠

We are almost to the point of proving the first part of Item 4. Here is the last step.

Lemma 9.5 *Suppose that $f : U \rightarrow \mathbf{C}$ is CA and $f'(a) \neq 0$ for some $a \in U$. Then, for all sufficiently small open sets V containing a , the set $f(V)$ is open.*

Proof: V just has to be small enough so that $f'(b) \neq 0$ for all $b \in V$. But then the previous result shows that $f(V)$ contains an open set that contains $f(b)$, for all $b \in V$. This proves that $f(V)$ is open. ♠