Transcendence of e by Rich Schwartz

I adapted this proof from the one in §5.2 of Herstein's *Topics in Algebra*. I think this proof is simpler and more businesslike.

The Main Step: Assume e is algebraic. Then e satisfies a polynomial equation with integer coefficients, having the following form.

$$\sum_{k=0}^{n} c_k e^k = 0; \qquad c_0 \neq 0; \qquad \max_k |c_k| < n. \tag{1}$$

Note that the degree of this equation might be less than n.

Below, we will produce an integer $p > n$ and a list $F(0), ..., F(n)$ of integers such that

1. $F(0)$ is not divisible by p.
2. $F(1), ..., F(n)$ are all divisible by p.
3. $|F(k) - e^k F(0)| < 1/n^2$ for $k = 1, ..., n$.

Note that

$$\sum_{k=0}^{n} c_k F(0) e^k = 0. \tag{2}$$

We're just multiplying Equation 1 by $F(0)$. Each term in the sum

$$\sum_{k=0}^{n} c_k F(k) \tag{3}$$

differs from the corresponding term in Equation 2 by at most $|c_k|/n^2$, which is less than $1/n$. Since the 0th terms agree and there are a total of $n+1$ terms, we see that in passing from Equation 2 to Equation3 we change the answer by less than 1. Since the sum in Equation 3 is an integer, this is only possible if the sum in Equation 3 is 0. Since $0 < |c_0| < n$, the quantity $c_0 F(0)$ is not divisible by p. The rest of the terms are divisible by p, so the whole sum in Equation 3 is not divisible by p. This is a contradiction.

Producing the List of Integers: It remains to produce the magic list of integers. Consider the function

$$F = \sum_{i=0}^{\infty} f^{(i)}; \qquad f(x) = \frac{x^{p-1}(1-x)^p(2-x)^p ... (n-x)^p}{(p-1)!}; \tag{4}$$

Here $f^{(i)}$ is the ith derivative of f. The sum for F is finite, because f is a polynomial. f is called a *Hermite polynomial*.

Property 1: We can write $f = ab$ where

$$a(x) = \frac{x^{p-1}}{(p-1)!}; \qquad b(x) = (1-x)^p \ldots (n-x)^p. \tag{5}$$

By the product rule for derivatives,

$$f^{(N)} = \sum_{i=0}^{N} C_{n,i} a^{(i)} b^{(N-i)}. \tag{6}$$

Here $C_{N,i}$ denotes N choose i. Note that $C_{N,0} = 1$. We have $a^{(p-1)}(0) = 1$ and otherwise $a^{(i)}(0) = 0$. Hence $f^{(p-1)}(0) = n!$ and $f^{(p)}, f^{(p+1)}$, etc. only involve terms whose coefficients are divisible by p. So, $F(0)$ is congruent to $n!$ mod p, and this number is not divisible by p.

Property 2: We can write $f = a \times b$, where

$$a(x) = \frac{(x-k)^p}{(p-1)!}; \qquad b(x) = x^{p-1}(1-x)^p \ldots \widehat{(k-x)}^p \ldots (n-x)^p. \tag{7}$$

The notation means that $(k-x)^p$ is omitted. We again have Equation 6. This time $a^{(p)}(k) = p$ and otherwise $a^{(i)}(k) = 0$. Hence $F(k)$ is divisible by p when $k = 1, 2, \ldots, n$.

Property 3: Let $\phi(x) = e^{-x} F(x)$. We compute

$$\phi'(x) = -e^{-x}(F(x) - F'(x)) = -e^{-x}\Big(\sum_{i=0}^{\infty} f^{(i)}(x) - \sum_{i=1}^{\infty} f^{(i)}(x)\Big) = -e^{-x} f(x).$$

The sums are finite, because f is a polynomial. Our equation tells us that $|\phi'(x)| \leq |f(x)|$ for $x \geq 0$. Hence

$$|F(k) - e^k F(0)| = |e^k||\phi(k) - \phi(0)| \leq k e^k \max_{[0,k]} |\phi'| \leq n e^n \max_{[0,n]} |f| \leq \frac{e^n (n^{n+2})^p}{(p-1)!}$$

For p sufficiently large, this last bound is less than $1/n^2$.