Math 181 Handout 11

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The purpose of this handout is work out some of the details of the Schwarz Christoffel Transformation. I will just work out the special case where the exponents are all $\pm 1/2$. The general case is very similar.

1 The Inverse Function Theorem

The material in this handout uses the Inverse Function Theorem, or IFT, in several places. Here is the statement:

Theorem 1.1 *Let $U \subset \boldsymbol{R}^2$ be an open set and suppose $F : U \to \boldsymbol{R}^2$ is a smooth map. If the differential dF is nonsingular at $p \in U$ then there exists a neighborhood V of p such that $f : V \to V' = F(V)$ is a diffeomorphism. Furthermore, the two linear matrices dF_p and $d(F^{-1})_{F(p)}$ are inverses of each other.*

The proof of this result can be found in any book on real analysis. In the case of complex analytic maps, dF is nonsingular if and only if $F'(z) \neq 0$.

Exercise 1: Suppose $U \subset \boldsymbol{C}$ is an open set and $F : U \to \boldsymbol{C}$ is complex analytic. Suppose $F'(p) \neq 0$ for some $p \in U$. Prove that there is a neighborhood V of p such that $F : V \to F(V)$ is a holomorphism. (Hint: You need to show that the inverse map is complex analytic.)

In the complex analytic case, the intuition behind the result is that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \text{H.O.T}$$

when $f'(z_0) \neq 0$. So, f basically looks like a linear map near such points.

2 The Schwarz-Christoffel Transformation

We're just going to consider a special case of the S-C Transformation. Suppose that $x_1 < x_2 < ... < x_n \in \boldsymbol{R}$ and $e_1, ..., e_n$ are numbers such that

- $e_j = \pm 1/2$ for $j = 1, ..., n$.
- $e_1 + ... + e_n = -2$.

Let $U \subset \boldsymbol{C}$ denote the upper halfplane. Let $U^* \subset \boldsymbol{C}$ denote the region obtained by deleting the closed downward pointing rays which start at $x_1, ..., x_n$. Then U^* is a simply connected open set which contains U and every point on $\boldsymbol{R}$ except $x_1, ..., x_n$. We are mainly interested in U, but the larger region U^* is convenient for for technical purposes.

Consider the function

$$f(z) = (z - x_1)^{e_1} ... (z - x_n)^{e_n}.$$

If we try to define this function in all of $\boldsymbol{C}$ we run into trouble in that we cannot consistently define f all the way around a loop which circles around x_j. It is not possible to make any loop like this in U^* and so f is defined and complex analytic in all of U^*.

We define

$$F(z) = \int f(z) dz.$$

This definition requires some explanation. We normalize so that $F(i) = 0$ and for any other point $z \in U^*$ we choose a smooth path γ connecting i to z and then define

$$F(z) = \int_\gamma f(z) dz.$$

This is well defined, independent of path, because the integral of f around any loop in U is zero.

Exercise 2: Prove that F is complex analytic in U^* and $F'(z) = f(z) \neq 0$. (Hint: Just write out the difference quotient and see what happens.)

Exercise 3: Prove that F extends continuously to each of $x_1, ..., x_n$ and also to ∞. (Hint: You just need to see that certain integrals are finite.)

3 The Image the Boundary

We want to understand $F(U)$, and it is useful to first understand $F(\boldsymbol{R}\cup\infty)$. Given Exercise 2, and the fact that F is defined on U^*, we know that F is defined on all of $\boldsymbol{R}$ and has a complex derivative at all points except $x_1,...,x_n$. These points naturally divide $\boldsymbol{R}$ into $n+1$ intervals. $I_0,...,I_n$. Actually, $I_0=(-\infty,x_1)$ and $I_n=(x_n,\infty)$ are rays.

When we square f we get

$$f^2(z)=(z-x_1)^{2e_1}...(Z-x_N)^{2e_n}.$$

Here $2e_j=\pm 1$. From this we see that f^2 is positive on I_n, negative on I_{n-1}, positive on I_{n-2}, and so on. This means that $f(z)$ is real for $z\in I_n$, pure imaginary for $z\in I_{n-1}$, real for $z\in I_{n-2}$, and so on. But $F'(z)=f(z)$, and the argument of $F'(z)$ tells us how F rotates points in a neighborhood of z. Hence $F(I_n)$ is a segment on a horizontal line; $F(I_{n-1})$ is a segment on a vertical line; $F(I_{n-2})$ is a segment on a horizontal line; and so on. Since F is continuous on $\boldsymbol{R}\cup\infty$, we see that these segments all piece together to give a closed zig-zag path, like the one shown in Figure 1. We've drawn an embedded example, but the curve doesn't have to be embedded.

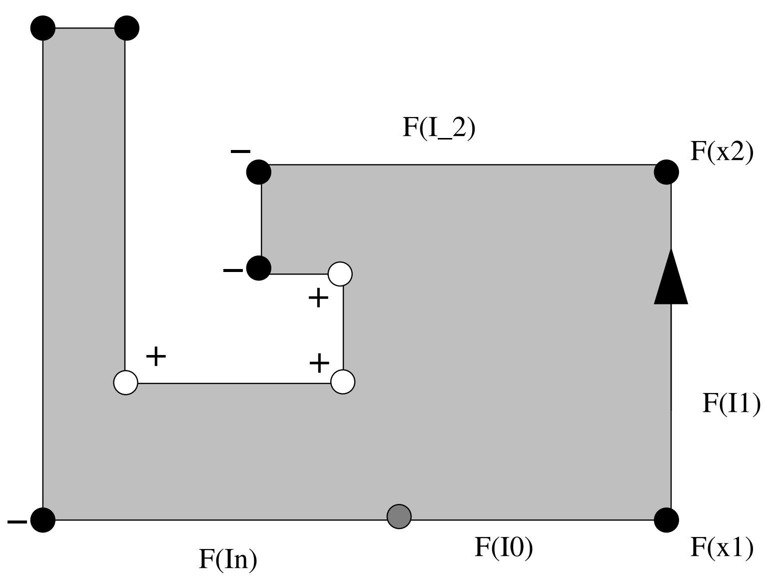

Figure 1

There are several fine points we need to make about this image.

- We orient $\boldsymbol{R}$ from $-\infty$ to $+\infty$. If you walk along $\boldsymbol{R}$ then U lies to your left. Being complex analytic, the map F is orientation preserving. This means that, as you walk around $F(\boldsymbol{R})$, the image $F(U)$ (at least locally) lies to the left.

- **Exercise 4:** Show that $F(\boldsymbol{R})$ turns left at x_j if $e_j = -1/2$ and right if $e_j = -1/2$. (See Figure 1.) Geometrically, $f(U)$ looks like one quadrant in a neighborhood of $f(x_j)$ if $e_j = -1/2$ and three quadrants if $e_j = 1/2$.

- **Exercise 5:** Given Exercise 4, and the fact that $e_1 + ... + e_n = -2$, show that $F(\boldsymbol{R})$ turns once around counterclockwise (the equivalent of 4 left turns.) Hence $F(I_0)$ and $F(I_n)$ travel in the same direction and fit together seamlessly.

4 The Image of the Upper Half-Plane

Let's say that F is *good* if $F(\boldsymbol{R} \cup \infty)$ has no self-intersections, as in Figure 1. Our goal is to prove

Theorem 4.1 *If F is good then F is a holomorphism (i.e. a complex analytic isomorphism) from U to the interior of the region bounded by $F(\boldsymbol{R} \cup \infty)$.*

We will prove this in several steps.

Lemma 4.2 *If F is good then $F(U)$ is contained in the interior of the region bounded by $F(\boldsymbol{R} \cup \infty)$.*

Proof: Let R be the region bounded by $F(\boldsymbol{R} \cup \infty)$. ($R$ is the shaded region shown in Figure 1.) If $F(U)$ is not contained in $\boldsymbol{R}$, then the closed and bounded set

$$K = F(U \cup \boldsymbol{R} \cup \infty)$$

contains some points which do not belong to R. In particular, there is a point $q = F(p) \in \partial K$ which does not belong to R. But $F'(p) \neq 0$. But then F maps a neighborhood of p to a neighborhood of q by the IFT. Hence $q \notin \partial B$. This contradiction shows that $F(U) \subset \boldsymbol{R}$. The same argument as we just gave shows that F cannot map an interior point of U onto a point of $\partial R = F(\boldsymbol{R} \cup \infty)$. ♠

Lemma 4.3 *F is one to one on $U \cup \boldsymbol{R} \cup \infty$.*

Proof: Let $B \subset U \cup \boldsymbol{R} \cup \infty$ denote the set of points where F is not one to one. In other words, if $z \in B$ then there is some other point z' such that $F(z) = F(z')$. We want to show that B is empty. We will suppose that B is nonempty and derive a contradiction.

Let
$$S = U \cup \boldsymbol{R} - x_1 - ... - x_n.$$
Then S is a connected subset of $\boldsymbol{C}$.

We claim that B is an open subset of S. Suppose that $z \in B$ and $F(z') = F(z)$. Note that F is one to one on $\boldsymbol{R} \cup \infty$. By the previous result, no point in $\boldsymbol{R} \cup \infty$ can be in B. Hence $z \in U$ and $z' \in U$. But $F'(z)$ and $F'(z')$ are nonzero. So, by the IFT, F maps little neighborhoods of z and z' onto little neighborhoods of $F(z) = F(z')$. Hence B contains a little neighborhood of z (and a little neighborhood of z'.) This shows that B is open.

We claim that B is closed subset of S. Suppose that $\{z_n\}$ is a sequence in B, converging to a point $z_\infty \in \partial B$. Consider the corresponding sequence $\{z'_n\}$. By taking a subsequence we can assume that $\{z'_n\}$ converges to some point z'_∞. By continuity we have $F(z_\infty) = F(z'_\infty)$. It suffices to show that $z_\infty \neq z'_\infty$. Note that F is complex analytic in a neighborhood of S and $F'(z) \neq 0$ for $z \in S$. In particular $F'(z_\infty) \neq 0$. But then F is one to one in a neighborhood of z_∞, by the IFT. Hence $z'_\infty \neq z_\infty$. This shows that $z_\infty \in B$. Hence B is closed in S.

Since B is nonempty, and open and closed in S, we must have $B = S$. Otherwise S would be disconnected. But B is disjoint from $\boldsymbol{R} - x_1 - ... - x_n$. Hence $B \neq S$. This contradiction shows that B is empty. ♠

Now we know that F^{-1} exists on all of R. But $F'(z) \neq 0$ on U. Hence F^{-1} is complex analytic by the IFT. This proves our theorem.

To restate the main result so far: If we fix $e_1, ..., e_n$ subject to the conditions above, and choose $x_1, ..., x_n$ in such a way that $F(\boldsymbol{R} \cup \infty)$ is a simple closed curve. Then F is a holomorphism from the upper half plane U onto the interior of the region R bounded by $F(\boldsymbol{R} \cup \infty)$.

5 A Special Case: Rectangles

Now consider the special case where $n = 4$. We can set things up so that

$$(x_1, x_2, x_3, x_4) = (-1, 0, r, 1),$$

with $r \in (0, 1)$. For each choice of r we get a S-C transformation F_r and a rectangular region R_r. We define the *first side* of R_r to be $F([-1, 0])$. We define the height of R_r to be the length of its first side. We define the width of R to be the length of the sides adjacent to the first side.

Let A_r denote the ratio of length to width of R_r. Then A_r measures the shape of R_r.

Lemma 5.1 *$A_r = A_s$ if and only if $r = s$.*

Proof: If $A_r = A_s$ then there is a map G of the form $G(z) = Mz + N$ such that F_r and $G \circ F_s$ have the same image. But then

$$H = (G \circ F_s)^{-1} \circ F_r$$

is a holomorphism from U to itself. But by the Disk Rigidity Theorem from the previous handout (also known as the Schwarz lemma) we know that H is a linear fractional transformation. But H fixes each of $-1, 0, 1$. A linear fractional transformation is determined by what it does to 3 distinct points. Hence H is the identity. In particular, $H(r) = r$. But $s = H(r)$. ♠

Exercise 6: Show that the function $r \to A_r$ is continuous.

Exercise 7: Now we know that the map $f(r) = A_r$ is an injective and continuous map from $(0, 1)$ to $(0, \infty)$. Prove that f is onto. In other words, you can find a holomorphism from U to any rectangle.

6 Generalizations

In the above discussion we always took $e_j = \pm 1/2$, but we did this just for simplicity. The value of e_j controlled the way that $F(\boldsymbol{R} \cup \infty)$ turned at $F(e_j)$. Here are some exercises which will take you through some other constructions, and which point the way towards the general construction.

Exercise 8: Find a holomorphism between U and the open region bounded by the triangle with angles α_1, α_2 and α_3. (Hint take $e_j = \alpha_j/\pi - 1$)

Exercise 9: (Challenge) Prove that there is a holomorphism between U and the interior of the region bounded by a regular pentagon. (Hint: Try first to get a general pentagon whose angles are all $3\pi/5$ and then use an intermediate value type argument to show that you can get the sides of the image to be all the same.

Exercise 10: (Challenge!) Let P be a region bounded by a simple closed polygon. Prove, using a combination of the SC transformation and a topological fixed point theorem, that there is a holomorphism from U to the interior of the region bounded by P. Once you have this result, you can get the Riemann mapping theory by polygonal approximation.