# Math 181 Handout 3

Rich Schwartz

September 15, 2005

The purpose of this handout is to define the fundamental group. Next week we will compute examples of the fundamental group. The fundamental group is a basic tool in topology and can be used to study surfaces. The structure of the handout is kind of funny. First, I'll talk about groups in general; then groups will disappear from the discussion for a while; then they'll come back in a really surprising way.

## 1 A Primer on Groups

It sounds like you guys have a pretty good grounding in groups, so I'll keep this section brief. If you haven't had any group theory, you can find a great treatment in e.g. Herstein's *Topics in Algebra*, vol 2.

A *group* is a set $G$, together with an "operation" $*$, which satisfies the following axioms.

- $g_1 * g_2$ is defined and belongs to $G$ for all $g_1, g_2 \in G$.
- $g_1 * (g_2 * g_2) = (g_1 * g_2) * g_3$ for all $g_1, g_2, g_3$.
- There exists a (unique) element $e \in G$ such that $e * g = g * e = g$ for all $g \in G$.
- For each $g \in G$ there is a (unique) element $h$ such that $g*h = h*g = e$. This element is called "$g$ inverse" and is usually written as $h = g^{-1}$.

The group $G$ is called *abelian* if, additionally, $g_1 * g_2$ and $g_2 * g_1$ are always equal. A *subgroup* of a group is a subset $H \subset G$ which is closed under the group law. So, if $h \in H$ then $h^{-1} \in H$ and if $h_1, h_2 \in H$ then $h_1 * h_2 \in H$.

Here are some examples of groups:

- $Z$, with the $+$ operation, forms an abelian group.
- If $G_1$ and $G_2$ are groups then $G_1 \times G_2$ can be made a group using the law $(g_1, g_2) * (h_1, h_2) = (g_1 * h_1, g_2 * h_2)$.
- The set $SL_n(\boldsymbol{Z})$ of $n \times n$ integer matrices with determinant 1 forms a non-abelian group.
- Let $A$ be a collection of $n$ things, for instance $A = \{1, ..., n\}$. Say that a *permutation* is a bijection $f : A \to A$. There are $n!$ different permutations, and they form a finite group. The $*$ operation is composition of maps. This group is called $S_n$.

Let $G_1$ and $G_2$ be groups. A map $f : G_1 \to G_2$ is a *homomorphism* if

$$f(a * b) = f(a) * f(b)$$

for all $a, b \in G_1$. Here the $*$ on the left is the rule for $G_1$ and the $*$ on the right is the one for $G_2$. The map $f$ is called an *isomorphism* if $f$ is a bijection and also a homomorphism. Here is a nice example. Let $G$ be a finite group and let $n$ be the number of elements in $G$. We're going to produce a homomorphism from $G$ into $S_n$, the permutation group on $n$ things. We're going to take the $n$ things to be the elements of $G$. So, given an element $g \in G$ how do we permute the elements of $G$? We define the map $f_g : G \to G$ using the rule $f_g(h) = gh$. It turns out that $f$ is a bijection, and $f_{g_1} = f_{g_2}$ only if $g_1 = g_2$. The map $g \to f_g$ is a one-to-one homomorphism from $G$ into $S_n$. This is *Cayley's theorem*: every finite group is isomorphic to a subgroup of a permutation group.

# 2 Homotopy Equivalence

Now we go back to metric spaces and manifolds. Let $X$ and $Y$ be metric spaces. Let $I = [0, 1]$ be the unit interval. Two maps $f_0, f_1 : X \to Y$ are said to be *homotopic* if there is a continuous map $F : X \times I \to Y$ such that

- $F(x, 0) = f_0(x)$ for all $x \in X$.
- $F(x, 1) = f_1(x)$ for all $x \in X$.

To explain the intuitive idea, it is useful to define $f_t : X \to Y$ by the formula $f_t(x) = F(x,t)$. Then the map $f_t$ interpolates between $f_0$ and $f_1$, with $f_t$ being very close to $f_0$ when $t$ is near 0 and $f_t$ being very close to $f_1$ when $t$ is near 1. The map $F$ is called *a homotopy* from $f_0$ to $f_1$.

If is useful to write $f_0 \sim f_1$ if these maps are homotopic. Let $C(X,Y)$ denote the set of all continuous maps from $X$ to $Y$. One can think of $\sim$ as a relation on the set $C(X,Y)$.

**Exercise 1:** Prove that $\sim$ is an equivalence relation on $C(X,Y)$.

**Exercise 2:** Prove that every two elements of $C(X, \boldsymbol{R}^n)$ are homotopic. Hint: Prove that any map $f : X \to \boldsymbol{R}^n$ is homotopic to the zero-map $f_0$ defined by the property $f_0(x) = 0$ for all $x$. Then, use the fact that $\sim$ is an equivalence relation.

**Exercise 3 (Challenge):** Let $P$ be a polynomial

$$P(x) = x^n + a_{n-1}x^{n-1} + ... + a_0.$$

Let $Q$ be the polynomial $Q(x) = x^n$. So, $P$ and $Q$ have the same leading term. We can think of $P$ as a map from the complex numbers $\boldsymbol{C}$ to the complex numbers. For any $R$ we can let $X \subset \boldsymbol{C}$ be the circle of radius $R$ centered at 0. That is

$$X = \{z \in \boldsymbol{C} |\ |z| = R\}.$$

First of all, prove that $0 \not\in P(X)$ if $R$ is sufficiently large. This means that we can think of $P$ and $Q$ as maps from $X$ to $Y = \boldsymbol{C} - \{0\}$. Prove that $P, Q : X \to Y$ are homotopic if $R$ is sufficiently large. This contrived-sounding problem is actually really important, as I'll explain in class.

# 3 The Fundamental Group

From now on we are going to take $X = I$, the unit interval, and we are going to study the space $Y$ by looking at the maps from $I$ into $Y$. For this entire discussion we choose a *basepoint* $y_0 \in Y$. This is just a special point we use throughout the construction. It turns out, in the end, that the final answers we get don't depend much on the choice of $y_0$.

Say that a *loop* in $Y$ is a continuous map $f : I \to Y$ such that

$$f(0) = f(1) = y_0.$$

The reason for the terminology should be pretty clear. Say that two loops $f_0$ and $f_1$ are *loop homotopic* if there is a homotopy $F$ from $f_0$ to $f_1$ such that $f_t$ is a loop for all $t \in [0,1]$. This is to say that $F(0,t) = F(1,t) = y_0$ for all $t$. We write $f_0 \sim f_1$ in this case. Just as in Exercise 1, this relation is an equivalence relation. Note that the equivalence relation here is slightly different than the one in the previous section, because of the added constraint that $F(0,t) = F(1,t) = y_0$ for all $t$.

As a set, $\pi_1(Y, y_0)$ is the set of equivalence classes of loops. The really interesting thing is that we can make $\pi_1(Y, y_0)$ into a group. Here's the construction. Suppose that we have two elements $[f]$ and $[g]$ of $\pi_1(Y, y_0)$. We can let $f$ and $g$ be representatives of the equivalence classes $[f]$ and $[g]$ respectively. That is, $f : [0,1] \to Y$ is a loop and $g[0,1] \to Y$ are both loops. We define the new loop $h = f * g$ by the following rule:

- If $x \in [0, 1/2]$ we define $h(x) = f(2x)$. That is, the first half of $h$ traces out all of $f$, but twice as fast.
- If $x \in [1/2, 1]$ we let $x' = x - 1/2$ and then we define $h(x) = g(2x')$. That is, the second half of $h$ traces out $g$, but twice as fast.

We write $h = f * g$.

**Exercise 4:** Suppose that $\widehat{f}$ and $\widehat{g}$ are different representatives for $[f]$ and $[g]$. That is $f$ and $\widehat{f}$ are equivalent loops and $g$ and $\widehat{g}$ are equivalent loops. Let $\widehat{h} = \widehat{f} * \widehat{g}$. Prove that $[\widehat{h}] = [h]$. In other words, prove that $h$ and $\widehat{h}$ are equivalent loops. This exercise is pretty easy, but quite tedious.

Given Exercise 4, we can define

$$[f] * [g] = [f * g]. \tag{1}$$

and this definition is independent of the equivalence class representatives we used to make the definition. This construction should remind you a bit about how one defines the group law on quotient groups. One needs to take coset representatives to make the definition, but then shows that the definition is independent of the choices.

**Exercise 5:** Show, for any three loops, $f, g, h$, that $(f * g) * h$ is equivalent to $f * (g * h)$. This means that $([f] * [g]) * [h] = [f] * ([g] * [h])$. This is the associative law for groups.

**Exercise 6:** Let $e$ be the loop defined by the rule $e(x) = y_0$ for all $x \in I$. Show that $[e] * [g] = [g] * [e] = [g]$ for all loops $g$. This means that $[e]$ plays the role of the identity element in $\pi_1(Y, y_0)$.

**Exercise 7:** Let $g$ be any loop. Define the loop $g^*$ by the formula $g^*(x) = g(1-x)$. In other words, $g^*$ traces out the same loop as $g$, but in the opposite direction. Prove the following result: If $g_1$ and $g_2$ are equivalent then $g_1^*$ and $g_2^*$ are equivalent. Finally, prove that $[g] * [g^*] = [e]$ and $[g^*] * [g] = [e]$. In other words, the inverse of $[g]$ is given by $[g^*]$.

Combining exercises 5, 6, and 7, we see that $\pi_1(Y, y_0)$ is a group. So, to each space $Y$ we can pick a basepoint $y_0$ and then define the group $\pi_1(Y, y_0)$. This group is known as the *fundamental group* of $Y$. (We'll see below that the group you get doesn't really depend on the base point.)

# 4 Changing the Basepoint

Say that two points $y_0, y_1$ are *connected by a path* if there is a continuous map $f : I \to Y$ such that $f(0) = y_0$ and $f(1) = y_1$. Say that $Y$ is *path connected* if every two points in $Y$ can be connected by a path. For instance $\boldsymbol{R}^n$ is path connected whereas $\boldsymbol{Z}$ is not.

**Lemma 4.1** *Suppose that $y_0, y_1 \in Y$ are connected by a path. Then $\pi_1(Y, y_0)$ and $\pi_1(Y, y_1)$ are isomorphic groups. In particular, if $Y$ is path connected then the (isomorphism type of the) group $\pi_1(Y, y)$ is independent of the choice of basepoint $y$ and we can just write $\pi_1(Y)$.*

**Proof:** (Sketch) Let $d$ be a path which joins $y_0$ to $y_1$. Let $d^*$ be the reverse path, which connects $y_1$ to $y_0$. We want to use $d$ and $d^*$ to define a map from $\pi_1(Y, y_0)$ to $\pi_1(Y, y_1)$. Given any $y_0$-loop $f_0 : I \to X$ with $f_0(0) = f_0(1) = y_0$ we can form a $y_1$-loop by the formula

$$f_1 = d * f * d_*.$$

In other words, the first part of $f_1$ travels backwards along $d$ from $y_1$ to $y_0$; the second part travels around $f_0$; and the third part travels back to $y_1$. You should picture a lasso in your mind.

Using arguments similar to the ones for the exercises above, you can show the following result: If $f_0$ and $\widehat{f}_0$ are equivalent, then $f_1$ and $\widehat{f}_1$ are equivalent. In other words, the map $H$, which sends $[f_0] \in \pi_1(Y, y_0)$ to $[f_1] \in \pi_1(Y, y_1)$ is well defined independent of the equivalence class representative used to define it. So, now we have a well defined map $H : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$. After this, one shows that $H$ is a homomorphism. That is, $H([f] * [g]) = H([f]) * H([g])$. This is not hard to do, once you draw a picture of what is going on.

Rather than show that $H$ is one-to-one and onto directly. One can define a map $H^* : \pi_1(Y, y_1) \to \pi_1(Y, y_0)$ just by reversing the roles of the two points. In other words, the loop $f_1$ is mapped to

$$f_0^* = d_* * f_1 * d.$$

Note that $f_0^*$ and $f_0$ are not precisely the same loop. If you draw pictures you will see that there is some extra "slack" in $f_0^*$. However, it turns out that $[f_0^*] = [f_0]$. In other words, the two loops are loop homotopic. Thus $H$ and $H^*$ are inverses of each other. Hence $H$ is an isomorphism (and so is $H^*$.) ♠

# 5 Functorality

The word *functoriality* refers to a situation where you are assigning one kind of an object to another in a way which respects the "natural" transformations between the two kinds of objects. This notion is defined precisely in any book on category theory. In our case, we are assigning a group $\pi_1(Y, y_0)$ to a *pointed space* $(Y, y_0)$. (By *pointed space* we mean a space with a chosen basepoint.) The natural transformations of pointed spaces are basepoint preserving continuous maps and the natural transformations between groups are homomorphisms.

We would like to see that our transformation (or *functor*) from spaces to groups respects these transformations. Here is Part I of the basic result:

**Lemma 5.1** *Let $(Y, y_0)$ and $(Z, z_0)$ be two pointed spaces and let $f : Y \to Z$ be a continuous map such that $f(y_0) = z_0$. Then there is a homomorphism $f_* : \pi_1(Y, y_0) \to \pi_1(Z, z_0)$.*

**Proof:** Let $[a] \in \pi_1(Y, y_0)$ be an equivalence class of loops, with representative $a$. So, $a : I \to Y$ is a loop. The composition $f \circ a$ is loop in $Z$. We define $f_*[a] = [f \circ a]$. If $[a_0] = [a_1]$ then there is a homotopy $H$ from $a_0$ to $a_1$. But then $f \circ H$ is a loop homotopy from $f \circ a_0$ to $f \circ a_1$. So, $[f \circ a_0] = [f \circ a_1]$ and our map is well defined. Note that $f \circ (a * b) = (f \circ a) * (f \circ b)$. Hence $f * ([a] * [b]) = (f_*([a])) * (f_*([b]))$. Hence $f_*$ is a homomorphism. ♠

Suppose that $f : Y \to Z$ is a continuous map and $g : Z \to W$ is a continuous map. Let's arrange so that $f(y_0) = z_0$ and $f(z_0) = w_0$. Then $g \circ f$ is a map from $Y$ to $W$ and $(g \circ f)_*$ is a homomorphism from $\pi_1(Y, y_0)$ to $\pi_1(W, w_0)$. Here is Part II of the basic result:

**Lemma 5.2** $(g \circ f)_* = g_* \circ f_*$.

**Proof:** Let $[a] \in \pi_1(Y, y_0)$. Then

$$(g \circ f)_*[a] = [(g \circ f) \circ a] = [g \circ (f \circ a)] = g_*[f \circ a] = g_* f_*[a].$$

That is it. ♠

If $f : Y \to Y$ is the identity map then $f_*$ is the identity map on $\pi_1(Y, y_0)$. Also, if $h : Y \to Z$ is a homeomorphism then we have the inverse homeomorphism $h^{-1}$. But $h \circ h^{-1}$ is the identity. Hence $h_* \circ h_*^{-1}$ is the identity homomorphism. Likewise $h^{-1} \circ h_*$ is the identity homomorphism. In short $h_*$ (and also $h_*^{-1}$) is a group isomorphism. So

**Theorem 5.3** *If $\pi_1(Y, y_0)$ and $\pi_1(Z, z_0)$ are <u>not</u> isomorphic groups then there is no homeomorphism from $Y$ to $Z$ which maps $y_0$ to $z_0$.*

The above is slightly contrived because we don't really care about these basepoints. Recall that $\pi_1(Y, y_0)$ doesn't depend on the basepoint if $Y$ is path connected. So

**Theorem 5.4** *Suppose $Y$ and $Z$ are path connected spaces. If $\pi_1(Y)$ and $\pi_1(Z)$ are not isomorphic then $Y$ and $Z$ are not homeomorphic.*

What's really great about this result is that we can use it to tell the difference between spaces just by looking at these groups. Of course, the question remains: How do we actually compute these groups. The next handout will go into a lot more detail on this.

## 6 Some First Steps

Here we'll just take some first steps in the computation of fundamental groups. Once we have more theory, these computations will be easy. So, what fundamental groups can we compute? It is easy to see (compare Exercise 2) that any two loops in $\boldsymbol{R}^n$ (based at 0) are equivalent. Hence $\pi_1(\boldsymbol{R}^n, 0)$ is the trivial group.

**Exercise 8A (Challenge):** Prove that there is a loop in $S^2$ (the two sphere) whose image is all of $S^2$. Hint: If you know about the Hilbert plane-filling curve from real analysis you're in good shape for this problem.

**Exercise 8B (Challenge)** Prove that $\pi_1(S^2, p)$ is the trivial group. Here $p \in S^2$ is any point. Hint: The intuitive idea is this: if the loop misses some point $q \neq p$, you can just "slide" the loop "down to $p$" by pushing it away from the missed point. However, you have to deal with the loops which come from Exercise 8A.

**Exercise 9:** If $(Y, y_0)$ and $(Z, z_0)$ are two pointed spaces then the product $(Y \times Z, (y_0, z_0)$ is again a pointed space. Prove that

$$\pi_1(Y \times Z, (y_0, z_0)) = \pi_1(Y, y_0) \times \pi_1(Z, z_0).$$

**Exercise 10: (Challenge)** Prove that $\pi_1(S^1, p)$ is nontrivial. Hint, think of $S^1$ as the unit circle in $\boldsymbol{R}^2$ and consider the loop

$$f(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Show that this loop is inequivalent to the identity loop.

Let $T = S^1 \times S^1$. Here $T^2$ is the torus. From Exercises 9 and 10 we know that $\pi_1(T^2)$ is nontrivial. (We don't worry about the basepoint because $T$ is obviously path connected.) On the other hand, by Exercise 8, $\pi_1(S^2)$ is trivial. Hence $S^2$ and $T^2$ are not homeomorphic!