# Math 181 Handout 6

Rich Schwartz

October 10, 2005

The purpose of this handout is to give you some background on smooth surfaces and Riemannian metrics. You can find this material in any beginning book on the theory of manifolds, such as Manfredo DoCarmo's book *Riemannian Geometry*.

This material is a prelude to the discussion of hyperbolic surfaces. In the first part of the handout I will define some basic objects in the Euclidean plane. In the second part of the handout I will explain how these ideas are transplanted onto a surface.

## 1 Curves in the Plane

A *smooth curve* in $\boldsymbol{R}^2$ is a smooth map $f:(a,b)\to\boldsymbol{R}^2$, given by equations

$$f(t)=(x(t),y(t))$$

such that $x(t)$ and $y(t)$ are smooth functions. This is to say that

$$\frac{d^n f}{dt^n}=\left(\frac{d^n x}{dt^n},\frac{d^n y}{dt^n}\right)$$

exists for all $n$. We will usually write $f'(t)$ for $df/dt$.

The function $f$ is *regular* if $f'(t)\neq 0$ for all $t\in(a,b)$. As usual $f'(t)$ is known as the *velocity* of $f$ at $t$. Sometimes it is useful to talk about smooth curves defined on closed intervals. Thus, if we write $f:[a,b]\to\boldsymbol{R}^2$ we really mean that $f$ is defined on some open interval $(a-\epsilon,b+\epsilon)$ and is smooth there. In particular $f:[0,0]\to\boldsymbol{R}^2$ is defined in a neighborhood of 0. This is the usual treatment of the problem with taking derivatives at the endpoints.

# 2 Inner Products

An *inner product* on a vector space $V$ is a map $G : V \times V \to \boldsymbol{R}$ which satisfies the following properties:

- $G(av + bw, x) = aG(v, x) + b(w, x)$. Here $a, b \in \boldsymbol{R}$ and $v, w, x \in V$.
- $G(x, y) = G(y, x)$.
- $G(x, x) \geq 0$ and $G(x, x) = 0$ if and only if $x = 0$.

You can remember this by noting that an inner product satisfies the same formal properties as the dot product.

Let's specialize to the case when $V$ is a 2 dimensional vector space, with a given basis $\{e_1, e_2\}$. Of course, you should think of $V = \boldsymbol{R}^2$ and $e_1 = (1, 0)$ and $e_2 = (0, 1)$, but it's worthwhile to do things more abstractly to allow for other possibilities. Given an inner product $G$ on $V$ we define

$$g_{ij} = G(e_i, e_j).$$

Then there is a symmetric $2 \times 2$ matrix

$$M_G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

which encodes $G$ completely.

**Exercise 1:** Let $v = a_1e_1 + a_2e_2$ and $w = b_1e_1 + b_2e_2$ be two vectors in $V$. Prove that

$$G(v, w) = \sum_{ij} g_{ij} a_i b_j.$$

Thus $G$ determines $M_G$ and *vice versa*.

**Exercise 2:** Give an example of a symmetric $2 \times 2$ matrix with $g_{11} > 0$ and $g_{22} > 0$ which goes not have the form $M_G$ where $G$ is an inner product. (You want to choose $g_{12}$ so the sum in Ex. 1 is sometimes negative.)

It is possible to explain exactly what conditions must be put on a symmetric matrix so that it comes from an inner product. The condition is simply that all the eigenvalues of the matrix are positive. A symmetric matrix with this property is called *positive definite*. Thus, the inner product $G$ determines a positive definite symmetric matrix $M_G$ and any positive definite symmetric matrix comes from an inner product.

# 3 Riemannian Metrics on the Plane

Let $\mathcal{I}$ denote the set of inner products on $\boldsymbol{R}^2$. Let $U \subset \boldsymbol{R}^2$ be an open set. A *Riemannian metric* on $U$ is a smooth map $\Psi : U \to \mathcal{I}$. In other words, a Riemannian metric on $U$ is a choice $G_p$ of inner product for each $p \in U$. This choice gives rise to the functions $g_{ij}(p)$. We require that the functions $g_{ij}$ are smooth functions on $U$. So, you can specify a Riemannian metric on $U$ by specifying 4 smooth functions $g_{ij} : U \to \boldsymbol{R}$ subject to the following constraints:

- $g_{12}(p) = g_{21}(p)$ for all $p \in U$.
- For all $p \in U$ the matrix $\{g_{ij}(p)\}$ is positive definite−i.e. has positive eigenvalues.

A *curve in* $U$ is just a curve which happens to lie entirely in $U$. We can measure the length of a curve in $U$ relative to the given Riemannian metric, as follows: Let $f : [a, b] \to U$ be a smooth curve. We define

$$\text{Riemannian length}(f) = \int_a^b \sqrt{G_{f(t)}(f'(t), f'(t))}\, dt.$$

The integrand above is called the *Riemannian speed* of $f$ at $t$. So, we are computing the Riemannian length of $f$ by integrating its Riemannian speed. Of course, these quantities depend on the choice of Riemannian metric. If we choose the standard Riemannian metric−namely the dot product at every point−then we recover the ordinary notions of speed and length.

**Exercise 3:** Let $U$ be the upper half plane in $\boldsymbol{R}^2$, namely those points $(x, y)$ with $y > 0$. Define

$$G_p(v, w) = \frac{v \cdot w}{y^2}.$$

Here $p = (x, y)$. In other words $G_p$ is just a multiple of the dot product at each point. The path $f(t) = (0, t)$, defined on $[1, 2]$ connects the point $(0, 1)$ to $(0, 2)$. Compute the Riemannian length of this path.

**Exercise 4:** Give an example of a Riemannian metric, defined on all of $\boldsymbol{R}^2$, which has the following property: Any two points in $\boldsymbol{R}^2$ can be joined by a smooth curve whose Riemannian length is less than 1.

# 4 Diffeomorphisms and Isometries

Let $U$ and $V$ be two open subsets of $\boldsymbol{R}^2$. A *diffeomorphism* from $U$ to $V$ is a homeomorphism $f : U \to V$ with the following additional properties:

- $f$ is smooth: That is, all orders of partial derivatives of $f$ exist.
- For each $p \in U$ the matrix $df(p)$ of first partial derivatives is non-singular. That is, $df$ defines a vector space isomorphism at each point. We abbreviate this by saying that $f$ is *regular*.
- $f^{-1}$ is smooth and regular.

Actually, the third condition follows from the other two and the inverse function theorem.

Note that $df_p$ maps a tangent vector based at $p$ to a tangent vector based at $f(p)$. Suppose that $U$ and $V$ are given Riemannian metrics. We say that a diffeomorphism $f : U \to V$ is a *Riemannian Isometry* if

$$H_{f(p)}(df_p(v), df_p(w)) = G_p(v, w); \qquad \forall v, w, p.$$

Here $v$ and $w$ are vectors and $p \in U$. Also $G$ is the Riemannian metric defined on $U$ and $H$ is the Riemannian metric defined on $V$.

**Exercise 6:** This problem refers to the problem in Exercise 3. Prove that the following maps are Riemannian isometries:

- $(x, y) \to (x + 1, y)$.
- $(x, y) \to (2x, 2y)$.
- $z \to -1/z$. Here we are using the complex notation $z = x + iy$.

The metric from Exercise 3 is known as the *Hyperbolic metric* and this exercise shows that it has a lot of isometries.

Here is another formulation of the notation of a Riemannian isometry: A Riemannian metric on $U \subset \boldsymbol{R}^2$ turns $U$ into a metric space, in the following way: Given $p, q \in U$ we define $S(p, q)$ to be the set of smooth curves in $U$ which join $p$ to $q$. We define $d(p, q)$ to be the infimum of the lengths of curves in $S(p, q)$.

**Exercise 7:** Prove that $d$ really is a metric on $U$. Prove also that a Riemannian isometry between $U$ and $V$ gives rise to a metric space isometry.

# 5 Atlases and Smooth Surfaces

Recall that a *surface* is a metric space $S$ such that every point has a neighborhood which is homeomorphic to $\boldsymbol{R}^2$. We say that a collection of such neighborhoods is called an *atlas*. The neighborhoods themselves are called *coordinate charts*. So, each element of the atlas is a pair $(U, h)$ where $U$ is an open subset of $\Sigma$ and $h : U \to \boldsymbol{R}^2$ is a homeomorphism. We require that the union of all the coordinate charts in the atlas is the entire surface. In other words, each point in the surface is contained in at least one coordinate chart.

Suppose now that $(U_1, h_1)$ and $(U_2, h_2)$ are two coordinate charts and it happens that $V = U_1 \cap U_2$ is not empty. We define $V_1 = h_1(V)$ and $V_2 = h_2(V)$. Being the intersection of two open sets, $V$ is an open subset of both $U_1$ and $U_2$. Since $h_1$ and $h_2$ are homeomorphisms, $V_1$ and $V_2$ are open subsets of $\boldsymbol{R}^2$. On $V_1$ the map

$$h_{12} = h_2 \circ h_1^{-1}$$

is well defined. We have $h_{12}(V_1) = V_2$. The map

$$h_{21} = h_1 \circ h_2^{-1}$$

is defined on $V_2$ and evidently $h_{21}(V_2) = V_1$. The two maps $h_{12}$ and $h_{21}$ are inverses of each other. Also, both maps are continuous, since they are the composition of continuous maps. In summary $h_{12} : V_1 \to V_2$ is a homeomorphism and $h_{21} : V_2 \to V_1$ is the inverse homeomorphism. These two functions are called *overlap functions* because they are defined on the overlaps between coordinate charts.

Our atlas on $\Sigma$ is said to be a *smooth structure* if all its overlap functions are smooth diffeomorphisms. In other words, every time we can produce an overlap function $h_{12} : V_1 \to V_2$ it turns out to be a diffeomorphism. We say that a *smooth surface* is a surface equipped with a smooth structure.

Here is an annoying technical point. Let $(U, h)$ be a pair such that $U$ is an open subset of $\Sigma$ and $h : U \to \boldsymbol{R}^2$ is a homeomorphism. If $(U, h)$ is not part of our atlas then we can enlarge our atlas by including $(U, h)$ in it. This will produce possibly some new overlap functions. If all the new overlap functions are diffeomorphisms then we say that $(U, h)$ is compatible with our atlas. We say that our atlas is *maximal* if it already contains all compatible coordinate charts. It is conventional for us to require that our atlases be maximal. However, this point never actually comes up in practice.

# 6 Smooth Curves and the Tangent Plane

Say that a map $f : (a, b) \to \Sigma$ is *smooth at* $t$ if there is some $\epsilon > 0$ such that

- $(t - \epsilon, t + \epsilon) \in (a, b)$;
- $f((t - \epsilon, t + \epsilon))$ is contained in a coordinate chart $(U, h)$ in our atlas.
- The curve $h \circ f : (t - \epsilon, t + \epsilon) \to \boldsymbol{R}^2$ is a smooth curve.

The fact that our overlap functions are all diffeomorphisms means that the notion of smoothness does not depend on which coordinate chart we use. In other words, if $f(t - \epsilon, t + \epsilon) \subset U_1 \cap U_2$ and $(U_1, h_1)$ and $(U_2, h_2)$ are both coordinate charts, then

$$h_2 \circ f = h_{12} \circ (h_1 \circ f).$$

Since $h_{12}$ is smooth, the curve $h_1 \circ f$ is smooth if and only if the curve $h_2 \circ f$ is smooth. Here are using the fact$-$really a consequence of the chain rule$-$that the composition of smooth maps is again smooth. (Any book on real analysis, or advanced several variable calculus, has this formulation of the chain rule.)

We say that $f : (a, b) \to \Sigma$ is *smooth* if $f$ is smooth at each $t \in (a, b)$. We say that $f : [a, b] \to \Sigma$ is smooth if $f$ is defined and smooth on a larger interval $(a - \epsilon, b + \epsilon)$.

Let $p \in \Sigma$ be a point. Suppose that

$$f_1, f_2 : [0, 0] \to \Sigma$$

are two curves wuch that $f_1(0) = f_2(0) = p$. We write $f_1 \sim f_2$ if there is a coordinate chart $(U, h)$ such that $p \in U$ and $h \circ f_1$ and $h \circ f_2$ have the same velocity at 0. In other words, $(h \circ f_1)'(0) = (h \circ f_2)'(0)$.

**Exercise 8:** Prove that $\sim$ is well defined, independent of the coordinate chart we use. Prove also that $\sim$ is an equivalence relation.

We define $T_p(\Sigma)$ to be the set of equivalence classes of curves $f : [0, 0] \to \Sigma$ such $f(0) = p$. We can make $T_p(\Sigma)$ into a vector space as follows: If $[f_1]$ and $[f_2]$ are two equivalence classes of curves, we define $[f_1] + [f_2]$ to be the equivalence class of the curve $g$ such that the velocity of $h \circ g$ is the velocity of $h \circ f_1$ plus the velocity of $h \circ f_2$. That is

$$(h \circ g)'(0) = (h \circ f_1)'(0) + (h \circ f_2)'(0).$$

**Exercise 9:** Prove that this notion of addition is well defined. In other words, if we made this definition relative to two different coordinate charts $(U_1, h_1)$ and $(U_2, h_2)$ then we could get the same answer. Hint: Use the fact that

$$h_2 \circ g = h_{12} \circ (h_1 \circ g)$$

(and likewise for $f_1$ and $f_2$) and the fact that $dh_{12}$ is a linear transformation at each point. Now use the chain rule.

We can also define scaling on $T_p(\Sigma)$. We define $r[f]$ to be the equivalence class of the curve which has $r$ times the velocity of $f$ at 0, measured in any coordinate chart. Again, this is well defined because the overlap functions are diffeomorphisms.

All in all, $T_p(\Sigma)$ is a vector space for each $p \in \Sigma$.

**Exercise 10:** Prove that $T_p(\Sigma)$ is isomorphic to $\boldsymbol{R}^2$. Hint: You should map $[f]$ to $(h \circ f)'(0)$ and show that it is an isomorphism.

# 7 Riemannian Surfaces

## 7.1 Basic Definition

Suppose that $\Sigma$ is a smooth surface. This means that we have a (maximal) atlas on $\Sigma$ whose overlap functions are smooth diffeomorphisms. Suppose, for each coordinate chart $(U, h)$, we choose a Riemannian metric on $\boldsymbol{R}^2$. We say that our choice is *consistent* if all the overlap functions are Riemannian isometries relative to the choices. Thus, the overlap function $h_{12}$ considered above is a Riemannian isometry from $V_1$ to $V_2$, when $V_1$ is equipped with the Riemannian metric associated to $(U_1, h_1)$ and $V_2$ is equipped with the Riemannian metric associated to $(U_2, h_2)$.

A *Riemannian metric* on $\Sigma$ is a consistent choice of Riemannian metrics on $\boldsymbol{R}^2$, one per coordinate chart. This definition is pretty abstract, so I'll give a second definition at the end of this section which is perhaps more concrete.

## 7.2 Riemannian Length

Let $f : [a, b] \to \Sigma$ be a smooth curve. We can define the *Riemannian Length* of $f$ as follows: First of all, we can find a partition $a = t_0 < ... < t_n = b$

such that $f([t_i, t_{i+1}])$ is contained in a coordinate chart $(U_i, h_i)$. Next, we can define $L_i$ to be the Riemannian length of

$$h_i \circ f([t_i, t_{i+1}]).$$

Finally, we define the length of $f$ to be $L_0 + ... + L_n$. In other words, we compute the lengths of a bunch of little pieces of $f$ and then add them together.

**Lemma 7.1** *The Riemannian length of $f$ is well defined, independent of the choices made in its definition.*

**Proof:** Suppose first of all that we keep the partition the same but use new coordinate charts $(U_i', h_i')$ such that $f([t_i, t_{i+1}]) \subset U_i'$. Then, on $[t_i, t_{i+1}]$ we have

$$h_i' \circ f = (h_i' \circ h_i) \circ (h_i \circ f).$$

But the map $h_i' \circ h_i$ is an overlap function, and is an isometry relative to the two Riemannian metrics. Thus $L_i = L_i'$. This shows that the Riemannian length of $f$ doesn't change if we use different coordinate charts from our atlas.

Suppose now that $a = s_0 < ... < s_m = b$ is another partition, and we are using a different sequence $\{(U_i', h_i')\}$ of coordinate charts to calculate the length. Then by considering all the $s_i$ and also all the $t_j$ (from our original partition) we can find a *refinement* $a = u_0 < ... < u_l = b$ which contains all the $s_i$ and also all the $t_j$. (Basically, we just take the collection of all the numbers and then resort them.)

We can use the charts $(U_i, h_i)$ to compute the length relative to the $u$-partition, and we will get the same answer as if we used the $t$-partition. The point here is just that integration is additive:

$$\int_{t_i}^{t_{i+1}} = \int_{t_1}^{u_{k+1}} + ... + \int_{u_{k+h-1}}^{t_{i+1}}.$$

Here $t_i = u_k < ... < u_{k+h} = t_{i+1}$. Likewise, we can use the charts $(U_i', h_i')$ to compute the length relative to the $u$-partition, and we will get the same answer as if we used the $s$-partition. Thus, we reduce to the case where the partition is the same but the charts change, considered previously. ♠

## 7.3 Another Point of View

Given that we have the notion of the tangent plane $T_p(\Sigma)$, and this object is always a vector space, we could define a Riemannian metric on $\Sigma$ to be a smoothly varying choice of inner product $G_p$ on $T_p(\Sigma)$ for each point $p \in \Sigma$. We just have to make sense of the notion of smoothness.

If we fix a coordinate chart $(U, h)$, then a Riemannian metric $G$ on $\Sigma$ gives rise to a Riemannian metric $H$ on $\boldsymbol{R}^2$ as follows. Suppose we have a point $q \in \boldsymbol{R}^2$ and two vectors $v, w$. Let $p = h^{-1}(q) \in U$ and $[f_1], [f_2] \in T_p(\Sigma)$ be the two classes so that $(h \circ f_1)'(0) = v$ and $(h \circ f_2)'(0) = w$. Then we define $H_q(v, w) = G_p([f_1], [f_2])$. To say that our Riemannian metric on $\Sigma$ varies smoothly is to say that $H$ is a smooth Riemannian metric on $\boldsymbol{R}^2$ for any choice of coordinate chart. This other definition is completely equivalent to the one I gave above.