Math 20 Final Solutions

A1: Write the function as $f(x,y) = xy\Delta$. Then $\nabla f = (y - 2x^2y, x - 4xy^2)\Delta$. Δ is never zero, so the critical points only occur when $y - 2x^2y = 0$ and $x - 4xy^2 = 0$. There are 5 solutions, namely $(0,0)$ and $(\pm 1/\sqrt{2}, \pm 1/2)$. Since the function tends to 0 as you go towards infinity, there must be a local min and a local max. Here is an analysis of the situation

- $f = 0$ at $(0,0)$.
- $f > 0$ at $(0, 1/\sqrt{2}, 1/2)$, and $(0, -1/\sqrt{2}, -1/2)$.
- $f < 0$ at $(0, -1/\sqrt{2}, 1/2)$, and $(0, 1/\sqrt{2}, -1/2)$.

Based on this analysis, the global max is $e^{-1}/(2\sqrt{2})$.

A2: We have the equation $r'' = (dv/dt)T + \kappa v^2 N = 2v^2 N$. The second equality comes from the fact that $dv/dt = 0$. From this we get $8 = \|r''\| = 2v^2$, which means that $v = 2$. So, the arc length is $2 \times 3 = 6$.

A3: In the (r,θ) plane, the domain is given by the following constraints: It satisfies $r \leq 1$ and $0 \leq \theta \leq \pi/4$. The last inequality comes from the inequality $\sin(\theta) \leq \cos(\theta)$. Compute the Jacobian:

$$J = \pm \det \begin{bmatrix} -3r\sin(\theta) & 3\cos(\theta) \\ 4r\cos(\theta) & 4\cos(\theta) \end{bmatrix} = 12r.$$

So, by change of variables, the area is

$$\int_0^{\pi/4} \int_0^1 12r \; dr \; d\theta = 3\pi/2.$$

A4: The tangent to the curve is proportional to the cross product of the two normals. This works out to $(1,2,1) \times (2x, 2y, 2z) = (A, B, 4x - 2y)$, where A and B are not important. The max/min height must occur where the tangent is horizontal, so $y = 2x$. Now we can use the first equation to solve for z, getting $z = 3 - 5x$. Plugging this into the second equation and solving yields $x = 0$ and $x = 1$. The two possible points are then $(0,0,3)$ and $(1,2,-2)$. The second one is obviously the lower point.

B1: Parametrize the circle as $r(t) = (\sqrt{2}\cos(t), \sqrt{2}\sin(t)$. The integral then becomes

$$A = \int_{\pi/4}^{3\pi/4} (\cos^4(t), 0) \cdot (-\sin(t), \cos(t))dt = -\int_{\pi/4} \cos^4(t)\sin(t)dt$$

Make the substitution $u = \cos(t)$ and $du = -\sin(t)dt$ to get the

$$A = \int_1^{-1} u^4 du = -2/5.$$

B2: This is a straight-up surface integral. Parametrize the surface using the equation $S(x, y) = (x, y, x^2 + y^2)$. Compute

$$N(x, y) = (1, 0, 2x) \times (0, 1, 2y) = (-2x, -2y, 1)$$

The integral is then

$$\int_D \int_D (x, 0, 2y(x^2 + y^2)) \cdot (-2x, -2y, 1).$$

Here D is the domain $x^2 + y^2 \leq 4$. The integral becomes

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} -2x^2 + 2x^2 y + 2y^3 \; dx \; dy.$$

B3: This vector field satisfies $Q_x = P_y$, and the domain has no holes. So, its conservative. Call the potential function f. We have $f_y = 2yx$. Integrating gives $f = y^2 x + g(x)$. Then $f_x = y^2 + g'(x) = 6x + y^2$. So, $g'(x) = 6x$. This gives $g(x) = 3x^2 + C$. So, $f = y^2 x + 3x^2 + C$, where C is any constant. The curl of the second vector field does not vanish, so its not conservative.

B4: Use Green's Theorem. The curl is $2x + 2y$, and so the integral is

$$2\int_{-1}^{0} \int_{x^2}^{-x} (x + y) \; dy \; dx = 2\int_{-1}^{0} (-x^2/2 - x^3 - x^4/2) = -1/30.$$

C1: Call the case $A = B = 0$ *the basic field*. The basic field is defined everywhere except $(0,0)$, and has divergence 0. The flux through any loop surrounding $(0,0)$ is the same and may be calculated using the unit circle. The result is: 2π. (This is the 2D Gauss law.) The flux through any loop that doesn't surround $(0,0)$ is 0. For general (A,B), the v.f. is a translate of the basic field, so you get the same result: the flux through any loop surrounding (A,B) is 2π and the flux through any other loop is 0. The circle in the problem surrounds the points $(0,0)$ and $(0,1)$ and $(1,0)$ and $(1,1)$, so for all these values of A and B you get flux 2π. Otherwise you get 0.

C2: Use Stokes' Theorem: The triangle in question has unit normal vector ($\vec{n} = (1/\sqrt{2}, -1/\sqrt{2}, 0)$ and F has been carefully constructed so that $\mathrm{curl}(F) \cdot \vec{n} = -\sqrt{2}$. The flux is constant, so the answer is just -(area of triangle) times $\sqrt{2}$. The triangle has area $\sqrt{2}/2$. So, the answer is -1.

C3: By the Divergence Theorem, the triple integral is the same as the flux of ∇f through the sphere. But the gradient of a function is always perpendicular to its level sets. So, $\nabla f \cdot n$ at each point is ± 3. The total flux is therefore ± 3 times the area of the sphere. The area of the sphere is 16π. So, the total flux is $\pm 48\pi$. Taking the absolute value, we get 48π for the final answer.

C4: By Gauss's law (and our choice of constants) the flux of any mass density through a membrane that surrounds it is -4π times the total mass. In our case, the total mass is 1, so the total gravitational flux through the donut is -4π. The whole picture is symmetric with respect to rotations about the z-axis, and also with respect to reflection in the xy plane. So, the amount of flux through T_2 is just 1/8 of the total flux, namely $-\pi/2$.