Math 20 Midterm 1 Solutions

1a. The velocity at $t = 1$ is $(1, 2, 3)$. The angle θ satisfies

$$\cos(\theta) = \frac{(1,0,0)\cdot(1,2,3)}{\|(1,0,0)\|\|(1,2,3)\|} = \frac{1}{\sqrt{14}}.$$

So $\sin^2(\theta) = 1 - 1/14 = 13/14$.

1b. The two lines are given by

$$(1,1,1) + t(1,2,3); \qquad (1,2,8) + t(1,4,12).$$

The lines are not parallel because the directions are not parallel. So, we just have to check if the lines intersect. The common normal to the two lines is

$$n = (1,2,3) \times (1,4,12) = (12,-9,2).$$

A vector pointing from one line to the other is given by the difference $V = (1,2,8) - (1,1,1) = (0,1,7)$. If the lines intersect, then V is perpendicular to n, but $V \cdot n = 5$, which is nonzero. So, the lines are skew.

2a. The function $f(x,y) = xy$ works because $f > 0$ when x and y have the sign and $f < 0$ when x and y have opposite signs. The function $f(x,y) = x^2 - y^2$ is another good example.
2b. The cheapest example if $f(x,y) = 0$, the zero function. Obviously the second derivative gives no information. A better example is something like $f(x,y) = x^4 + y^4$. For this second example, $(0,0)$ is the absolute minimum, and all second partials of f are 0 at $(0,0)$.

3. The chain rule gives

$$\frac{d}{dt} f(r(t))|_{t=0} = r'(0) \cdot \nabla(0) = r'(0) \cdot 1,3,3) = \sqrt{19}\; v\cos(\theta).$$

Here $\sqrt{19}$ is the norm of $\nabla f(0)$ and θ is the angle between the velocity $r'(0)$ and $\nabla f(0)$. We want to make v as small as possible and have $\sqrt{19}\; v\cos(\theta) = 1$. Since $|\cos(\theta)| \le 1$, the best we can do is make the curve go in the direction of $\nabla f(0)$, which means setting $\theta = 0$, and then taking $v = 1/\sqrt{19}$. So, the answer is $1/\sqrt{19}$.

4. Here is the cheapest solution. Let L be the line $y = x - 1$. Let $p = (x, x^2)$ be the point closest to the line $y = x - 1$. The tangent line to the parabola at p must be parallel to L, and hence have slope 1. But the slope of the tangent line is $2x$. Setting $2x = 1$ gives $x = 1/2$. So, the point must be $(1/2, 1/4)$.

Here is another solution. The parabola lines on one side of L and the function $F(x, y) = x - y$ is proportional to the function that measures distance to L. The Lagrange multiplier equation $\nabla F = \lambda \nabla g$ gives

$$(1, 1) = \lambda\ (1, 2x),$$

from which you again get $x = 1/2$.

Here is another solution. Parametrize the parabola by (x, x^2) and parametrize the line by $(y, y - 1)$. Then you want to minimize the function

$$F(x, y) = ((x - y)^2 + (x^2 - y + 1)^2.$$

Use the first derivative test:

$$F_x = 2(x - y) + 4x(x^2 - y + 1) = 0.$$

$$F_y = -2(x - y) - 2(x^2 - y + 1) = 0.$$

Add the two equations together, to get

$$(4x - 2)(x^2 - y + 1) = 0.$$

One solution is clearly $x = 1/2$. The other possibility is $x^2 - y + 1 = 0$, or $y = x^2 + 1$. But in this case, $F_x = 0$ then leads to $x = y$. So, here, $x = x^2 + 1$. This has no real solutions. So, only $x = 1/2$ works. This is probably the hardest way to to the problem.