Math 52 Sample Midterm 2
(The solutions are on the second page.)

1. Find the equation of the plane in R^3 that consists of points which are equidistant to the point $(0,5,3)$ and the point $(-2,3,1)$.

2.Recall that a *rotation* of $\boldsymbol{R}^4$ is a map that preserves the dot product. Prove that there is a rotation of $\boldsymbol{R}^4$ that maps the vector $(1,2,3,4)$ to the vector $(5,0,1,2)$. (You don't need to write out an explicit formula.)

3. Recall that the span of a set S is the set of all finite linear combinations of elements of S. Prove that

$$\text{Span}(\text{Span}(S)) \subset \text{Span}(S)$$

for any subset S of vectors in a vector space.

4. Let $\{v_1, ..., v_n\}$ be a basis for a finite dimensional vector space V. Let $\{w_1, ..., w_n\}$ be another basis. We can take each v_k and write it as a linear combination

$$v_k = a_{k1}w_1 + ... + a_{kn}w_n.$$

In this way we get an $n \times n$ matrix $A = \{a_{ij}\}$. Prove that A is invertible.

Solutions:

1. Let $v_1 = (0, 5, 3)$ and let $v_2 = (-2, 3, 1)$. The vector pointing from v_1 to v_2 is the vector $v_3 := v_2 - v_1 = (-2, -2, -2)$. The vector pointing from the v_1 to the midpoint of v_1 and v_2 is $v_4 := v_1 + \frac{1}{2}v_3 = (-1, 4, 2)$. This is one of the points on the plane. Any other point on the plane has the form $v_4 + w$, where w is perpendidular to v_3. This is to say that a point (x, y, z) lies in the plane if and only if

$$((x, y, z) - v_4) \cdot v_3 = 0.$$

This works out to be $x + y + z = 5$.

2. Let $v_1 = (1, 2, 3, 4)$ and $w = (5, 0, 1, 2)$. Note first that $v_1 \cdot v_1 = w \cdot w = 30$, so we would expect this problem to work out. Let T_1 be the map

$$T_1(x_1, x_2, x_3, x_4) = (x_3, x_4, x_1, x_2).$$

Then clearly T_1 preserves the dot product. Also $T_1(v_1) = (3, 4, 1, 2)$. Since the composition of two rotations−i.e. the effect of doing one first and the second one−is also a rotation, it now suffices to find a rotation that maps $(3, 4, 1, 2)$ to $(5, 0, 1, 2)$. You can check explicitly that a linear transformation represented by the matrix

$$\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is a rotation as long as $c^2 + s^2 = 1$. (Here c and s are really abbreviations for $\cos(\theta)$ and $\sin(\theta)$.) To get what we want, we take $c = 3/5$ and $s = 4/5$. Then the resulting linear transformation T_2 has the property that $T_2(3, 4, 1, 2) = (5, 0, 1, 2)$. So, all in all, the rotation T_3 defined by the equation $T_3(x) = T_2(T_1(x))$ maps v_1 to w.

3. An arbitrary element of Span(Span(S)) has the form

$$x = a_1 v_1 + ... + a_n v_n$$

where $v_1, ..., v_n \in \text{Span}(S)$ and $a_1, ..., a_n$ are real numbers. By definition each v_j has for the form

$$v_j = b_{j1} w_1 + ... + b_{jm} w_m,$$

where $w_1, ..., w_m$ is some finite list of vectors in Span(S). (Note: Even though the different v's might be linear combinations of different vectors in S, we can just make one master list of w's that contains all the vectors used in any of the combinations−there will just possibly be a lot of 0s in the equations above.) So, now we can write

$$x = a_1(b_{11}w_1 + ... + b_{1m}w_n) + ... + a_n(b_{n1}w_1 + ... + b_{nm}w_m) = c_1w_1 + ...c_mw_m,$$

where the c's are real numbers obtained by expanding everything out and grouping terms. This shows that $x \in$ Span(S).

4. Let M_1 be the matrix obtained by writing the basis vectors $v_1, ..., v_n$ as columns of a square matrix. Let M_2 be the matrix obtained by writing the basis vectors $w_1, ..., w_n$ as columns of a square matrix. From the very definition of matrix multiplication, $M_1A^t = M_2$. (Here A^t is the transpose of A.) For instance, the first column of M_1A^t is $a_{11}v_1 + ... + a_{1n}v_n = w_1$. Since $v_1, ..., v_n$ is a basis, the matrix M_1 is invertible. Likewise M_2 is invertible. So, we can write

$$A^t = M_1^{-1}M_2,$$

the product of invertible matrices. Hence A^t is invertible. This means that A^t has nonzero determinant. But A and A^t have the same determinant. So, A is also invertible.