Handout 1

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The purpose of this handout is to discuss some notions in geometry. If some or even most of this handout makes no sense, you shouldn't panic or think that this seminar isn't for you. The handout is meant as a reference, and we're going to be talking a lot about the objects discussed in the handout, as well as how to read text like this. I hardly knew any of the stuff when I started college.

At the end of each section I'll pose one or two questions. If you're comfortable with the material in a given section, you might want to think about the questions. If the material in the section doesn't make that much sense, you should just ignore the questions.

1 The Euclidean Plane

The Euclidean plane is usually denoted by $\boldsymbol{R}^2$, where $\boldsymbol{R}$ stands for the set of real numbers. Points in the Euclidean plane are usually denoted by (x, y) or else (x_1, x_2), etc.

Writing $p = (x_1, x_2)$ and $q = (y_1, y_2)$, the distance between the points p_1 and p_2 is given by

$$d(p, q) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}. \tag{1}$$

This is the usual formula.

An *isometry* of $\boldsymbol{R}^2$ is a map $T : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ such that

$$d(T(p), T(q)) = d(p, q) \tag{2}$$

for all points $p, q \in \boldsymbol{R}^2$. By *map*, I mean a rule that assigns a point $T(p)$ for each point p. For example, the map $T(x, y) = (x + 1, y)$ is an isometry.

Geometrically, this map picks up the plane and translates it one unit to the right. As another example, the map

$$(x,y) \to (x\cos(\theta) + y\sin(\theta), -x\sin(\theta) + y\cos(\theta)) \tag{3}$$

is an isometry for any choice of θ. The map $T(x,y) = (2x,y)$ is not an isometry.

Questions:

- Can you prove that the map in Equation 3 is really an isometry?
- Can you figure out all the isometries of the plane?

2 Euclidean Space

In general, n dimensional Euclidean space is denoted by $\boldsymbol{R}^n$, and points in $\boldsymbol{R}^n$ are denoted by $(x_1, ..., x_n)$. Points in $\boldsymbol{R}^n$ are sometimes called *vectors*. You will learn a lot about $\boldsymbol{R}^n$ in a linear algebra course.

Note: Whenever you are reading about something like $\boldsymbol{R}^n$, you should usually have in mind a simple example, like $\boldsymbol{R}^3$. You can much better understand the general case if you knew a few simple examples really well.

The distance between two points $p = (x_1, ..., x_n)$ and $q = (y_1, ..., y_n)$ is given by

$$d(p,q) = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}. \tag{4}$$

An isometry of Euclidean space is defined just as for the Euclidean plane.

The *dot product* between two vectors in $\boldsymbol{R}^n$ is given by

$$p \cdot q = x_1 y_1 + ... + x_n y_n. \tag{5}$$

The *norm* of p is given by

$$\|p\| = \sqrt{p \cdot p} \tag{6}$$

With this notation, the distance formula above reads

$$d(p,q) = \sqrt{\|p - q\|}. \tag{7}$$

A *hyperplane* in Euclidean space is the set of solutions to an equation of the form

$$p \cdot v = C. \tag{8}$$

Here v is a nonzero vector in $\boldsymbol{R}^n$ and C is a constant. That, the hyperplane consists of all the points p that satisfy the equation. Some special cases:

- For $n = 1$, a hyperplane is just a point.
- For $n = 2$, a hyperplane is a straight line.
- For $n = 3$, a hyperplane is a copy of the plane.

A *flat* is any finite intersection of hyperplanes. For instance, if two different hyperplanes in $\boldsymbol{R}^3$ intersect, then they intersect in a line. Similarly, three hyperplanes in $\boldsymbol{R}^3$ can intersect in a single point. So, the flats in $\boldsymbol{R}^3$ are points, lines, and planes. A flat has a *dimension*, which I won't define formally. Informally, a flat is a copy of $\boldsymbol{R}^k$, and the dimension is k.

Questions:

- Can you give a definition of dimension?
- Suppose that you have two flats in $\boldsymbol{R}^n$ that have the same dimension. Can you prove that there is an isometry of $\boldsymbol{R}^n$ that maps one flat to the other.

3 Convexity

As you read this section, you should think about the simplest cases first, even though the text might be talking about something in general. So, I'm going to talk about what convexity means in $\boldsymbol{R}^n$, but you should draw pictures in $\boldsymbol{R}^2$. The pictures in $\boldsymbol{R}^1$ are even simpler, but they're pretty boring!

A subset $S \subset \boldsymbol{R}^n$ is *convex* if, for any two points $p_1, p_2 \in S$, the line segment joining p_1 to p_2 also lies in S. The line segment I mean can be described as the set of points of the form

$$tp_1 + (1-t)p_2, \tag{9}$$

with t in the interval $[0, 1]$. A line segment is a subset of a 1 dimensional flat.

Here are some examples.

- A disk is a convex subset of $\boldsymbol{R}^2$.
- Triangles and squares are convex.
- Some quadrilaterals are not convex. (Draw a picture of one.)
- A solid cube is convex.
- A solid ellipsoid is convex.
- The intersection of any number of convex sets is again convex.

Given any subset $S \subset \boldsymbol{R}^n$, the *convex hull* of S, sometimes denoted by Hull(S) is defined to be the intersection of all the convex sets that contain S. So, the convex hull of S is the smallest convex set that contains S.

A *convex polytope* in $\boldsymbol{R}^n$ is the convex hull of a finite set of points, provided that the convex hull is genuinely n-dimensional. For instance, the convex hull of 2 points in $\boldsymbol{R}^n$ is always a line segment, but we wouldn't really want to call this line segment a polytope because it is just 1-dimensional. Polytopes are called *xonvex polygons* in $\boldsymbol{R}^2$ and *convex polyhedra* in $\boldsymbol{R}^3$.

Questions:

- Suppose that S is a collection of 5 points in $\boldsymbol{R}^2$. What are the possibilities for the convex hull of S?
- Suppose that S is a collection of 5 points in $\boldsymbol{R}^3$. What are the possibilities for the convex hull of S?
- Write the letters of your name in the plane. Think of them as sets. Sketch their convex hulls.
- What is the convex hull of a hyperbola?
- Suppose you are given a set of 100 points in $\boldsymbol{R}^2$. Can you think of a way to systematically compute the convex hull of the set? This is a computer-science type question.
- Suppose that P is a polytope in $\boldsymbol{R}^n$ and Π is a hyperplane in $\boldsymbol{R}^n$ that divides P into two pieces. Prove that each piece is again a polytope? (Draw a picture in $\boldsymbol{R}^2$.)

4 Groups

A *group* is a set G, together with an operation for combining elements of G. The operation, sometimes denoted by a $(*)$, has to satisfy 4 axioms in order for it to turn G into a group. Here are the axioms.

1. For any $g_1, g_2 \in G$, the operation $g_1 * g_2$ makes sense as an element of G. In other words, the operation "always works".

2. $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$ for all g_1, g_2, g_3. This is sometimes called *the associative law.*

3. There is an element of g, called e, such that $g * e = e * g = g$ for all g. In other words, e is kind of like the "zero element".

4. For any g there is an element h such that $g*h = h*g = e$. The element h is sort of like "negative g". Sometimes h is written as g^{-1}.

You will learn a lot about groups when you take abstract algebra (Math 152).

Here are some examples:

- $G = \boldsymbol{Z}$, the integers, and $*$ is just $+$. In this case $e = 0$ and $g^{-1} = -g$.

- G is the set of hours on the clock and $*$ is just "clock addition". For instance,
$$8:00 * 9:00 = 5:00. \tag{10}$$
because it is $5:00$ nine hours after it is $8:00$. Here $e = 12:00$. What is $(8:00)^{-1}$?

- You can imagine making the same construction with a clock that has n hours. The resulting group is called $\boldsymbol{Z}/n$.

- The set of isometries of $\boldsymbol{R}^n$ is a group. The $(*)$ law is composition. In other words, the map $T_1 * T_2$ has the action
$$T_1 * T_2(p) = T_1(T_2(p)). \tag{11}$$
The isometry that does nothing–i.e., the identity map–is e. The isometry T^{-1} just "undoes" whatever T does. For instance, if T rotates $\boldsymbol{R}^2$ clockwise by 5 degrees, then T^{-1} rotates $\boldsymbol{R}^2$ counterclockwise by 5 degrees.

One nice way to make groups is to let P be a polytope in $\boldsymbol{R}^n$ and consider the set G of all isometries T such that $T(P) = P$. For instance, if P is a square in $\boldsymbol{R}^2$ then you can rotate T by 90, degrees, 180 degrees, etc. This group is often called the *symmetry group* of P.

Questions:

- It turns out that the group of symmetries of a square has 8 different elements. Can you draw what they all do?
- How many elements does the group of symmetries of a cube have?
- The n-dimensional cube is the convex hull of the set of all points of the form $(x_1, ..., x_n)$, where each coordinate is either 0 or 1. How many elements does the symmetry group of the n-dimensional cube have?
- Any polytope P always has the identity map as an element of its symmetry group? Can you draw a polygon in the plane that only has this element in its symmetry group?
- Some groups have the property that $g * h = h * g$ for all g and h. These groups are called *abelian*. Prove that the symmetry group of the square is not abelian?
- Can you draw a polygon that has an abelian symmetry group?