Fundamental Domains and Orbifolds

Rich Schwartz

September 24, 2007

The purpose of this handout is to discuss the notions of fundamental domains and orbifolds. Each of these topics presents some expository difficulties.

- There isn't really a universal definition for a *fundamental domain*, because it is a broad notion used in many contexts. I will give a typical definition, for the context of planar symmetry groups.
- The formal definition of an *orbifold* is quite tricky. I will first give an informal definition. Then, I'll give a formal definition. The formal definition is much harder to understand!

1 Discrete Symmetry Groups

Let $\boldsymbol{R}^2$ denote the Euclidean plane. Let $\boldsymbol{EUC}$ denote the group of isometries of $\boldsymbol{R}^2$. So, each element of $\boldsymbol{EUC}$ is a symmetry of the plane that preserves distances. The group $\boldsymbol{EUC}$ is sometimes called the *Euclidean group*. Rather than specify a symmetry pattern, we will specify a subset $G \subset \boldsymbol{EUC}$ that is a group in its own right. We think of G as the group of symmetries of some unspecified infinite tiling. G is called a *subgroup* of $\boldsymbol{EUC}$, because G is both a subset and a group.

We call G a *discrete symmetry group* if it has the following property: For any disk B in the plane, there are only finitely many elements $g \in G$ such that $g(B) \cap B$ is not empty. In other words, all but finitely many elements of G move B completely off itself.

Here is an example. Let G denote the group of elements having the form

$$g(x, y) = (x, y) + (m, n),$$

with m and n integers. In other words, G is the group consisting of all the integer translations. If B is a disk, then there is some N such that all points in B have coordinates of the form (x, y) with $|x|$ and $|y|$ both less than N. In other words, B is contained in a square of side length $2N$ centered at the origin. But then $g(B) \cap B = \emptyset$ as long as g translates at least one of the coordinates by more than $2N$.

Here is a non-example. Let G be as above, except that m and n can be rational numbers. Then there are infinitely many elements of G such that $g(B) \cap B \neq \emptyset$ for any choice of B. The point is that there is an infinite list of elements of G that move points of the plane less than any tiny amount you like.

Exercise 1: Let T be the tiling of the plane by unit squares having vertices at integer coordinates. Let G' be the group of symmetries of T. Prove that G' is a discrete group. Note that G' is larger than our example G, because G includes (for example) a 90 degree rotation about the origin.

2 Orbits

Let G be a discrete symmetry group. Given any point $p \in \boldsymbol{R}^2$ the orbit of p, denoted $G(p)$ is the set

$$G(p) = \{g(p) | \; g \in G\}. \tag{1}$$

In other words, you act on p by every single element of g and then take the totality of points you get. If G is the example considered above, then the orbit of the point $(0, 0)$ is exactly the set of points with integer coordinates.

Exercise 2: Let G' be the group considered in Exercise 1. Draw the orbit $G'(1/2, 1/3)$.

We can relate the notion of discreteness to the notion of orbits. A group G is discrete if and only if every orbit of G intersects any disk in finitely many points. In other words, when you look at the orbits of a discrete group, you see a "discrete" set of points, spaced out sort of like as in a grid.

3 Solid Polygons

Say that a *polygonal arc* is a curve in the plane made from finitely many line segments, $I_1, ..., I_n$, with the following properties:

- I_k and I_{k+1} share a common vertex for each $k = 1, ..., n-1$.
- I_k and I_m are disjoint if $|k - m| > 1$.

A *polygon* is the same kind of object, except that I_1 and I_n share a common vertex. Figure 1 shows some examples.

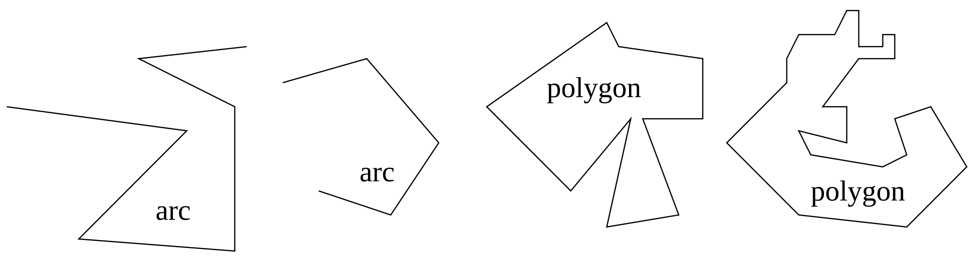

Figure 1

Say that a *solid polygon* is a region in the plane whose boundary is a polygon. In other words, you draw the polygon and then fill in the bounded region. It is a theorem, a special case of the Jordan Curve Theorem, that any polygon divides the plane into two connected regions, one of which is what we call a solid polygon. Try finding coloring in the solid polygon in Figure 2.

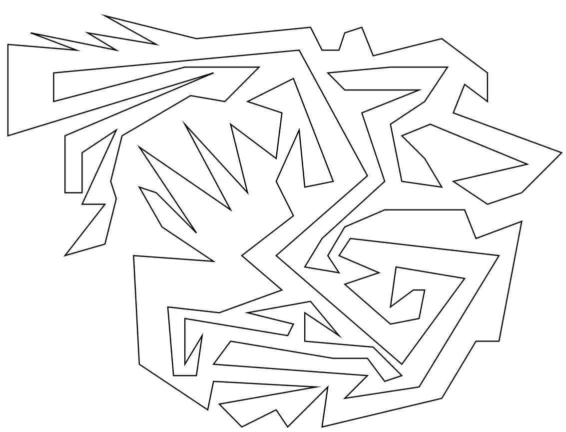

Figure 2

3.1 Fundamental Domains

Let G be a discrete symmetry group, as above. A *fundamental domain* for G is a solid polygon F with two properties:

- Every orbit of G intersects F in at least one point.
- An orbit that intersects the interior of F only intersects F in one point.

The *interior* of F is the set of points not lying on the polygon that bounds F. Put another way, a point x lies in the interior of F is a sufficiently small disk centered at x also lies in F.

Exercise 3: Draw fundamental domains for the examples G and G' discussed above.

The symmetry group does not uniquely determine the fundamental domain. That is, the same symmetry group can have many different fundamental domains. The beauty of some of the Escher tilings comes from Escher's pleasing choice of fundamental domain for underlying symmetry group.

Lemma 3.1 *Every symmetry group G has a convex fundamental domain.*

Proof: If every point of $\boldsymbol{R}^2$ was fixed by some element of G, then G would not be discrete. So, there is at least one point $p \in \boldsymbol{R}^2$ that is not fixed by any element of G. Consider the orbit $G(p)$. Let F denote the set of points in $\boldsymbol{R}^2$ that are at least as close to p as they are to any other point in $G(p)$. We want to verify the two axioms for F, and we also want to see that F is convex.

Axiom 1: Let q' be some other point in $\boldsymbol{R}^2$. We want to show that $G(q')$ intersects F in at least one point. There is some point $p' \in G(p)$ such that q' is as close to p' as it is to any other point in $G(p)$. Also there is some element g such that $g(p) = p'$. But then let $q = g^{-1}(q')$. The distance from q to p is the same as the distance from q' to p'. Also, the minimum distance from q to $G(p)$ is the same as the minimum distance from q' to $G(p)$, because g maps $G(p)$ to itself. Hence p is the point of $G(p)$ closest to q. Hence $q \in F$. Since $q \in G(q')$, we see that $G(q')$ intersects F in at least one point.

Axiom 2: Suppose that x is a point in the plane and $g_1, g_2 \in G$ are elements such that $g_1(x) \in F$ and $g_2(x) \in F$. Consider the sets

$$F_1 = g_1^{-1}(F); \qquad F_2 = g_2^{-1}(F),$$

and the points

$$p_1 = g_1^{-1}(p); \qquad p_2 = g_2^{-1}(p),$$

Then F_1 consists of those points in $\boldsymbol{R}^2$ that are as close to p_1 as they are to any other point in $G(p)$. Likewise, F_2 consists of those points in $\boldsymbol{R}^2$ that are as close to p_2 as they are to any other point in $G(p)$. Being in both F_1 and F_2, we see that x is equidistant from p_1 and p_2.

We claim that $p_1 \neq p_2$. suppose that $p_1 = p_2$. Then $g_1^{-1}(p) = g_2^{-1}(p)$. But then $g_1 g_2^{-1}(p) = p$. But then $g_1 = g_2$. This contradiction shows that $p_1 \neq p_2$. So, x is equidistant from the two distinct points, p_1 and p_2.

If x_1 lies in the interior of F then x lies in the interior of F_1. But then we can move x a little bit towards p_2 and away from p_1 to a new point y. The new point y still belongs to F_1, but y is closer to p_2 than to p_1. this is a contradiction. The contradiction shows that x_1 does not lie in the interior of F. The same argument shows that x_2 does not lie in the interior of F.

Convexity: Let $q_1, q_2, q_3...$ be a complete list of points of $G(p)$ other than p. Let H_n be the set of points that are at least as close to p as to q_n. Then H_n is a halfplane. Hence H_n is convex. But then F is the intersection of all the sets H_n. The intersection of convex sets is convex. Hence F is convex. ♠

4 Gluing Diagrams

Let G be a discrete symmetry group and let F be a fundamental domain. Let ∂F denote the boundary of F. Say that two points x and y in ∂F are *equivalent* if they belong to the same orbit. That is, $G(x) = G(y)$. Note that ∂F is already a polygon. It turns out that ∂F can be further divided into smaller segments (if necessary) so that the segments are matched in pairs. In other words, for each segment S_1, there is another segment S_2, having the same length, such that each point on S_1 is equivalent to a point on S_2. The equivalent points are in the same positions on the two segments. In other words, if p_1 on S_1 is halfway between the endpoints of S_1, then the equivalent point $p_2 \in S_2$ is halfway between the two endpoints of S_2.

The pairing of segments is denoted by arrows. For instance, for the group G discussed above, we could take F to be a unit square. Then the pairings are as shown in Figure 3. The decoration of F with these arrows is called the *gluing diagram*.

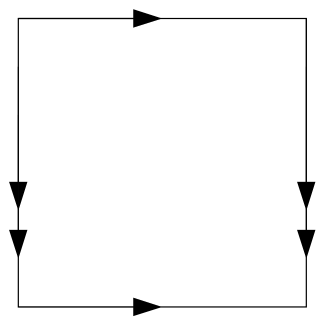

Figure 3

Exercise 4: Find a fundamental domain for the group G', and then find the gluing diagram for the fundamental domain you have found.

5 Orbifolds

Informally, a *Euclidean orbifold* is the space one gets from a pair (G, F) by cutting F out of the plane and gluing the sides of ∂F together according to the gluing diagram. Here G is a planar symmetry group and F is a fundamental domain for G. what is misleading about this informal definition is that the orbifold associated to (G, F) only depends on G and not on F. In other words, if you took a different fundamental domain F' for the same group and then did the (new) gluings, you would get the same orbifold.

More formally, a Euclidean orbifold O_G for the group G is the space of orbits of G, equipped with the *quotient metric*. To say that O_G is the *space of orbits* of G is to say that there is one point of O_G for each orbit of G. If you think about this, the space we get by gluing F together according to the gluing diagram accomplishes exactly this goal. To say that O_G is *equipped with the quotient metric* is a bit harder to explain. What follows is a terse account of the basic notions.

A *metric* on a set S is a function $d : S \times S \to \boldsymbol{R}$ with the following properties:

- $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$.
- $d(x,y) = d(y,x)$ for all $x, y \in X$.
- $d(x,y) + d(y,z) \leq d(z,y)$ for any $x, y, z \in S$.

For instance, the function $d(x,y) = \|x - y\|$ is the usual Euclidean metric on the plane.

A *ball* in a metric space is a set of the form

$$\{y | d(y,x) = r\}$$

for some choice of $x \in S$ and some $r > 0$. We call x the *center* of the ball. In Euclidean spaces, the balls are the usual round balls. However, in a general metric space, the balls can be pretty wierd.

An *isometry* between metric spaces X and Y is a map $f : X \to Y$ such that

$$d_Y(f(p), f(q)) = d_X(p, q)$$

for all $p, q \in X$. We also insist that f is one-to-one and onto. If we drop the one-to-one condition, then the map f is called an *isometric submersion*.

To say that O_G is equipped with the quotient metric is to say that there is a map $\phi : \boldsymbol{R}^2 \to O_G$ with the following properties:

- For each $x \in \boldsymbol{R}^2$, the point $\phi(x)$ is the point in O_G corresponding to the orbit of x.
- If x is not fixed by any element of G, then the map ϕ is an isometry from some ball centered at x to some ball centered at $\phi(x)$.
- If x is fixed by some element of G, then ϕ is an isometric submersion from some ball centered at x to some ball centered at $\phi(x)$.