Mobius Transformations and Circles

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The purpose of this handout is to prove that Mobius transformations map circles to circles.

1 Basic Definition and Group Structure

A *Mobius transformation* is a map of the form

$$f(x) = \frac{Az+B}{Cz+D}; \qquad AD - BC = 1. \tag{1}$$

In class we considered the case when A, B, C, D are all real numbers, but we will also consider the case when A, B, C, D are any complex numbers. First we will verify that the Mobius transformations form a group using the composition law.

Exercise 1: Suppose that f_1 and f_2 are Mobius transformations. Prove that $f_1 \circ f_2$ is also a Mobius transformation. Here $f_1 \circ f_2(z) = f_1(f_2(z))$.

A rather tedious, but routine calculation, shows that

$$f_1 \circ (f_2 \circ f_3) = (f_1 \circ f_2) \circ f_3.$$

This fact has a conceptual explanation. Each Mobius transformation is represented by a 2×2 matrix. Composition of the Mobius transformations corresponds to multiplication of the matrices. Matrix multiplication satisfies the associative law, and therefore so does the composition of Mobius transformations.

If we define the map

$$g(z) = \frac{Dz - B}{-Cz + A}, \tag{2}$$

then $f \circ g(z) = z$ and $g \circ f(z) = z$. In other words, g is the inverse of f. What is going on here is that the matrices corresponding to f and g, namely

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}; \qquad \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$$

are inverses of each other.

Now we have all the ingredients to say that the Mobius transformations form a group. Exercise 1 shows that the first axiom holds. The associative lat is the second axiom. The map $f(z) = z$ is the identity element. This is the third axiom. Finally, inverses exist. This is the fourth axiom.

2 The Riemann Sphere

The Riemann sphere is defined to be the set $\boldsymbol{C} \cup \infty$. Here $\boldsymbol{C}$ is the set of complex numbers, and ∞ is considered to be an extra point.

We really need the ∞ symbol to be included if we want to have our Mobius transformations everywhere defined. For instance, if $f(z) = 1/z$, then we want to say that $f(0) = \infty$ and $f(\infty) = 0$.

In general, if you have a Mobius transformation f, you define

$$f(\infty) = \lim_{n \to \infty} f(n). \tag{3}$$

For instance, suppose that

$$f(z) = \frac{2z + 1}{3z + 2}.$$

Then

$$f(\infty) = \lim_{n \to \infty} \frac{2n + 1}{3n + 2} = \lim_{n \to \infty} \frac{2 + \frac{1}{n}}{3 + \frac{2}{n}} = \frac{2}{3}.$$

After thinking about it for a minute, you see that

$$\lim_{n \to \infty} \frac{An + B}{Cn + D} = \frac{A}{C}. \tag{4}$$

We think of ∞ as lying on every straight line. If L is such a straight line, then we call $L \cup \infty$ a *circle*.

3 Triples of Points

Say that a *triple* is a triple of points of the form (z_1, z_2, z_3) where z_1, z_2, z_3 all belong to $\boldsymbol{C} \cup \infty$, and

$$z_1 \neq z_2 \neq z_3 \neq z_1.$$

That is, the points are all distinct.

Here is the main result in this section.

Theorem 3.1 *Let (z_1, z_2, z_3) and (z_1', z_2', z_3') be triples. Then there is a unique Mobius transformation T such that $T(z_1, z_2, z_3) = (z_1', z_2', z_3')$.*

The notation $T(z_1, z_2, z_3) = (z_1', z_2', z_3')$ means that $T(z_1) = z_1'$, etc.

We will prove the result by stringing together a number of smaller results.

Lemma 3.2 *If (z_1, z_2, z_3) is any triple, then there is a Mobius transformation T such that $T(z_1, z_2, z_3) = (0, 1, \infty)$.*

Proof: Step 1: Let

$$T_1(z) = \frac{1}{z - z_3}.$$

Then $T_1(z_3) = \infty$. Let $w_1 = T_1(z_1)$ and $w_2 = T_1(z_2)$. So, T_1 maps the triple (z_1, z_2, z_3) to the triple (w_1, w_2, ∞).
Step 2: Let

$$T_2(z) = z - w_1.$$

Note that $T_2(\infty) = \infty$ and $T_2(w_1) = 0$. Let $u_2 = w_2 - w_1$. Then T_2 maps the triple (w_1, w_2, ∞) to $(0, u_2, \infty)$.
Step 3: Let

$$T_3(z) = \frac{z}{u_2}.$$

Then $T_3(\infty) = \infty$ and $T_3(u_2) = 1$ and $T_3(0) = 0$. So, T_3 maps the triple $(0, u_2, \infty)$ to $(0, 1, \infty)$.
Step 4: If we let $T = T_3 \circ T_2 \circ T_1$ then

$$T(z_1, z_2, z_3) = T_3(T_2(T_1(z_1, z_2, z_3))) =$$

$$T_3(T_2(w_1, w_2, \infty)) = T_3(0, u_2, \infty) = (0, 1, \infty).$$

That's the end of the proof. ♠

Lemma 3.3 *Suppose* (z_1, z_2, z_3) *and* (z_1', z_2', z_3') *are any two triples. There is a Mobius transformation that maps* (z_1, z_2, z_3) *to* (z_1', z_2', z_3').

Proof: By the previous result, there is a Mobius transformation T such that $T(z_1, z_2, z_3) = (0, 1, \infty)$. Likewise, there is a Mobius transformation T' such that $T'(z_1', z_2', z_3') = (0, 1, \infty)$. Let $S = (T')^{-1} \circ T$. Then

$$S(z_1, z_2, z_3) = (T')^{-1}(T(z_1, z_2, z_3)) =$$

$$(T')^{-1}(0, 1, \infty) = (z_1', z_2', z_3').$$

The point here is that T' maps (z_1', z_2', z_3') to $(0, 1, \infty)$, to the inverse map $(T')^{-1}$ maps $(0, 1, \infty)$ to (z_1', z_2', z_3'). So, S is the Mobius transformation that does the job for us. ♠

So far, we've worked out the existence of a Mobius transformation that maps (z_1, z_2, z_3) to (z_1', z_2', z_3'). The next step is to show that it is unique.

Lemma 3.4 *Suppose that* T *is a Mobius transformation with the property that* $T(0, 1, \infty) = (0, 1, \infty)$. *Then* T *is the identity. That is,* $T(z) = z$.

Proof: Let's write

$$T(z) = \frac{Az + B}{Cz + D}.$$

Since $T(0) = 0$ we must have $B = 0$. Since $T(\infty) = \infty$ we must have $C = 0$. Now we know that $T(z) = (A/D)z$. But $T(1) = 1$. Therefore, $A = D$. Since $AD - BC = 1$, we must have either $A = D = -1$ or $A = D = 1$. Both cases lead to the same map $T(z) = z$. ♠

Now for the end of the proof. Suppose that $T_1(z_1, z_2, z_3) = (z_1', z_2', z_3')$. Likewise suppose that $T_2(z_1, z_2, z_3) = (z_1', z_2', z_3')$. Let U be a Mobius transformation such that $U(z_1', z_2', z_3') = (0, 1, \infty)$. Let $S_1 = U \circ T_1$ and $S_2 = U \circ T_2$. Note that S_1 and S_2 both map (z_1, z_2, z_3) to $(0, 1, \infty)$. Therefore $S_1^{-1} \circ S_2$ maps $(0, 1, \infty)$ to $(0, 1, \infty)$. Therefore $S_1^{-1} \circ S_2$ is the identity. But $S_1^{-1} \circ S_1$ is also the identity. So $S_1^{-1} \circ S_1 = S_1^{-1} \circ S_2$. But this means that $S_1 = S_2$. Since $S_1 = S_2$ we have $U \circ T_1 = U \circ T_2$. But then $T_1 = T_2$.

In our last argument we used the *cancellation* property of groups several times: If $AB = AC$ then $B = C$.

4 Real Triples

Say that a triple (z_1, z_2, z_3) is *real* if each of the points is either a real number of else ∞. For instance $(0, 1, \infty)$ is a real triple. Say that a Mobius transformation is *real* if the coefficients defining it, namely (A, B, C, D), are all real numbers.

Exercise 2: Prove the following theorem, which is almost exactly like what we proved in the last section: If (z_1, z_2, z_3) and (z_1', z_2', z_3') are real triples, then there is a unique real Mobius transformation T such that $T(z_1, z_2, z_3) = (z_1', z_2', z_3')$. (Hint: Just take all the arguments in the previous section and put the word *real* before each occurance of the word *mobius transformation*.)

5 Closed Loops

The set $\boldsymbol{R} \cup \infty$ is a special example of a circle in the Riemann sphere. Say that a *closed loop* is a set of the form $M(\boldsymbol{R} \cup \infty)$ where M is a Mobius transformation. We want to prove that closed loops are actually circles. Here we will sketch the proof of a preliminary result about these closed loops.

Lemma 5.1 *Suppose that X is a closed loop, but not a circle. Then there is a Mobius transformation T such that $T(X)$ and X intersect in at least 3 points, but $X \neq T(X)$. In fact, T can be taken as a rotation.*

Proof: Figure 1 shows the idea of this proof. Let $\boldsymbol{X}$ be the region in the plane bounded by X. Suppose X is not a circle. Then $\boldsymbol{X}$ is not a disk. Let D_1 be any choice of the largest possible disk contained in $\boldsymbol{X}$. Then D_1 intersects X in at least 2 points. Otherwise, we could make D_1 larger. Likewise, let D_2 be any choice of the smallest possible disk containing $\boldsymbol{X}$. Then D_2 intersects X in at least 2 points. Let D_3 be any disk that contains D_1 and is contained in D_2 but is not equal to either one. Let Y be the circle that bounds D_3. Then Y intersects X in at least 3 points, but $Y \neq X$. The point here is that any arc of X that travels from the boundary of D_1 to the boundary of D_2 must touch Y. If T is a small rotation about Y then $T(X)$ and X intersect in at least 4 points, but are unequal. These rotations are all Mobius transformations. They all have the form $T(z) = uz + v$ where u is a unit complex number. ♠

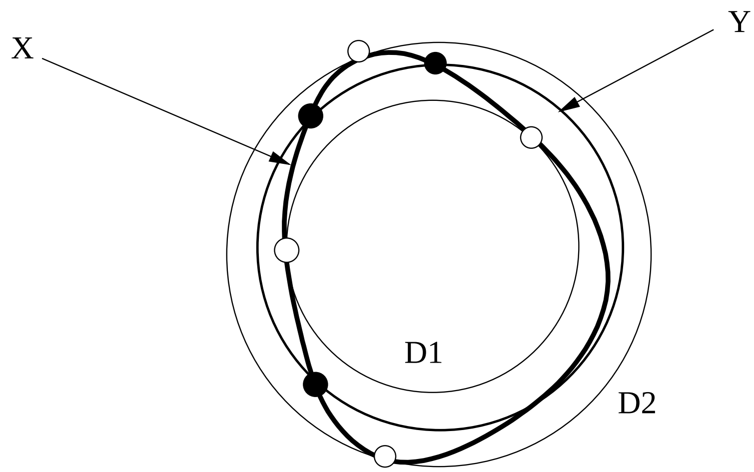

Figure 1: Loops and Circles

6 The Image of the Reals

Here is the main result in this section.

Theorem 6.1 *If M is any Mobius transformation, then $M(\boldsymbol{R}\cup\infty)$ is a circle. Also, if $\boldsymbol{C}$ is any circle in $\boldsymbol{C}$, then there is some Mobius transformation T such that $T(\boldsymbol{R}\cup\infty) = C$.*

In the above theorem, we mean the generalized sense of the word *circle*, in which $L\cup\infty$ counts as a circle when L is a straight line.

Lemma 6.2 *Suppose that $M(\boldsymbol{R}\cup\infty)$ contains ∞. Then $M(\boldsymbol{R}\cup\infty)$ has the form $L\cup\infty$, where L is a straight line.*

Proof: Let's write

$$M(z) = \frac{Az+B}{Cz+D}.$$

If $M(\infty) = \infty$, then $C = 0$. This means that $M(z) = (A/D)z + (B/D)$. This map is just a dilation followed by a translation. In this case, $M(\boldsymbol{R})$ is clearly just another line. All we are doing is expanding the picture, rotating it, and translating it. So, $M(\boldsymbol{R}\cup\infty) = L\cup\infty$ for some straight line L.

Let's suppose that $M(\infty) \neq \infty$. Then there is some point $t \in \boldsymbol{R}$ such that $M(t) = \infty$. But then there is a real Mobius transformation T such that $T(\infty) = t$. Consider the map $M' = M \circ T$. Since T is a real Mobius transformation, we have $T(\boldsymbol{R} \cup \infty) = \boldsymbol{R} \cup \infty$. This means that

$$M'(\boldsymbol{R} \cup \infty) = M(T(\boldsymbol{R} \cup \infty)) = M(\boldsymbol{R} \cup \infty).$$

But $M'(\infty) = \infty$. By the previous case, we see that $M'(\boldsymbol{R} \cup \infty) = L \cup \infty$, where L is a straight line. But $M'(\boldsymbol{R} \cup \infty) = M(\boldsymbol{R} \cup \infty)$. ♠

Lemma 6.3 *Suppose that $M(\boldsymbol{R} \cup \infty)$ does not contain ∞. Then $M(\boldsymbol{R} \cup \infty)$ is a circle.*

Proof: Let $X = M(\boldsymbol{R} \cup \infty)$. Suppose that X is not a circle. By Lemma 5.1, there is some Mobius transformation T such that $T(X)$ and X intersect in 3 points, but $T(X) \neq X$.

Let (z_1, z_2, z_3) be a triple that is contained in both X and $T(X)$. Consider the two maps $M_1 = M$ and $M_2 = T \circ M$. Note that $M_1(\boldsymbol{R} \cup \infty) = X$ and $M_2(R \cup \infty) = R(X)$.

Let (a_1, a_2, a_3) be the real triple such that $M_1(a_1, a_2, a_3) = (z_1, z_2, z_3)$. Likewise, let (b_1, b_2, b_3) be the triple such that $M_2(b_1, b_2, b_3) = (z_1, z_2, z_3)$. There is a real Mobius transformation S such that $S(a_1, a_2, a_3) = (b_1, b_2, b_3)$. Consider the maps

$$M_1' = M_1; \qquad M_2' = M_2 \circ S.$$

Note that M_1' and M_2' both map (a_1, a_2, a_3) to (z_1, z_2, z_3). From our uniqueness theorem above, we see that $M_1' = M_2'$. But

$$M_2'(\boldsymbol{R} \cup \infty) = M_2 \circ S(\boldsymbol{R} \cup \infty) = M_2(\boldsymbol{R} \cup \infty) = T(X).$$

On the other hand

$$M_1'(\boldsymbol{R} \cup \infty) = M_1(\boldsymbol{R} \cup \infty) = X.$$

Since $M_1' = M_2'$ we must have $X = T(X)$. This contradicts the fact that $T(X) \neq X$. ♠

Now we know that $M(\boldsymbol{R} \cup \infty)$ is always a circle, in the generalized sense. In other words, we count $L \cup \infty$ as a circle when L is a straight line.

Lemma 6.4 *Let C be any circle in $\boldsymbol{C}$, the complex plane. Then there is a Mobius transformation M such that $M(\boldsymbol{R} \cup \infty) = C$.*

Proof: Consider the Mobius transformation

$$T(z) = \frac{1}{z - i}.$$

Then $T(i) = \infty$. Hence $T(\boldsymbol{R} \cup \infty)$ does not contain ∞. Let $C' = T(\boldsymbol{R} \cup \infty)$. We know that C' is some circle in $\boldsymbol{C}$, but perhaps not the one we want.

We can find a Mobius transformation S such that $S(C') = C$. This just amounts to rotating, dilating, and translating the plane. But then let $M = S \circ T$. We have $M(\boldsymbol{R} \cup \infty) = S(C') = C$. ♠

7 The End of the Proof

Now we can finish the proof that Mobius transformations map circles to circles. Suppose that M is a Mobius transformation and C is a circle. We want to prove that $M(C)$ is a circle. We can find a Mobius transformation T such that $T(\boldsymbol{R} \cup \infty) = C$. Consider $M' = M \circ T$. We have

$$M'(\boldsymbol{R} \cup \infty) = M(C).$$

But $M'(\boldsymbol{R} \cup \infty)$ is a circle, by the result in the previous section. So, $M(C)$ is circle.