Basics of Hyperbolic Geometry

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The purpose of this handout is to explain some of the basics of hyperbolic geometry. I'll talk entirely about the hyperbolic plane.

1 The Model

Let $\boldsymbol{C}$ denote the complex numbers. The *hyperbolic plane*, as a set, consists of the complex numbers $x + iy$, with $y > 0$. This set is denoted by $\boldsymbol{H}^2$. The set $\partial \boldsymbol{H}^2$ is another name for the complex number of the form $x + 0i$. In other words, $\partial \boldsymbol{H}^2$ is just the real line.

The *geodesics* in $\boldsymbol{H}^2$ are either circles that meet $\partial \boldsymbol{H}^2$ at right angles. The vertical rays count as "circles". See Figure 1.

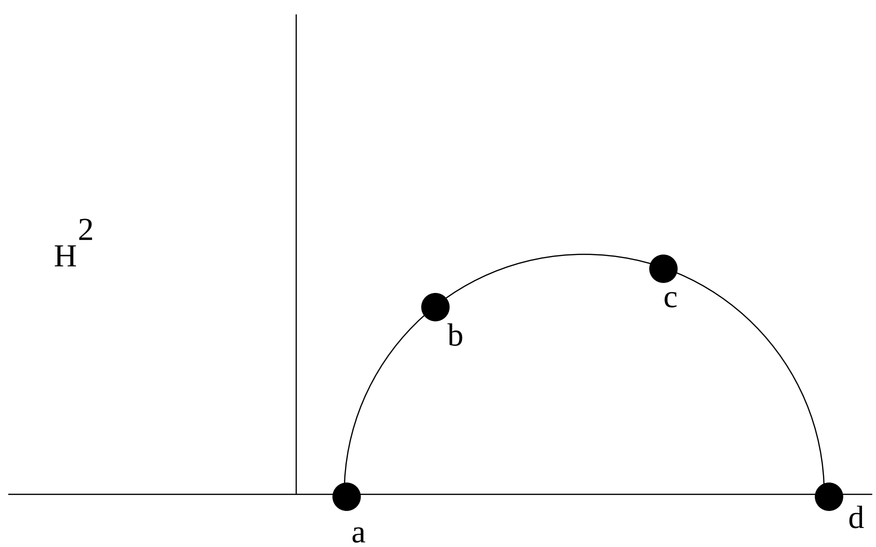

Figure 1: Geodesics and Points

Exercise 1: Prove that any two distinct point in $\boldsymbol{H}^2$ determine a unique geodesic that contains these two points.

Exercise 2: Let γ be any geodesic in $\boldsymbol{H}^2$ and let p be any point in $\boldsymbol{H}^2$ not on γ. prove that there are infinitely many geodesics through p that do not contain γ. This is the failure of the parallel postulete for hyperbolic geometry.

The *angle* between any two geodesics, at a point of intersection, is defined as usual as the angle between two curves: It is the angle of intersection between the tangent lines to the circles. A *geodesic segment* is a circular arc contained in a geodesic.

Exercise 3: Let p_1, p_2, p_3 be three distinct points in $\boldsymbol{H}^2$. These points naturally determine a *geodesic triangle*. You can connect each of the points to the other two by geodesic segments. Prove that the sum of interior angles in any triangles is less than 180 degrees.

2 The Distance Formula

Given two points $b, c \in \boldsymbol{H}^2$, we can find the points a, d such that b and d are the endpoints of the geodesic joining a to b, as in Figure 1. If b and c are not on the same vertical line then we define

$$\Delta(b,c) = \log \frac{|a-c||b-d|}{|a-b||c-d|}. \tag{1}$$

If b and c lie on the same vertical line then one of a or d is ∞. In this case (assuming that $d = \infty$), we have

$$\Delta(b,c) = \log \frac{|a-c|}{|a-b|}. \tag{2}$$

Exercise 4: Prove that $\Delta(b,c) \geq 0$ and $\Delta(b,c) = 0$ if and only if $b = c$.

Exercise 5: Explain how Equation 2 is a limiting case of Equation 1. Put another way, prove that $\Delta(b,c)$ is a continuous function of the locations of b and c.

3 Symmetries of the Model

A *Mobius transformation* is a map of the form

$$f(x) = \frac{Az+B}{Cz+D}; \qquad AD - BC = 1. \tag{3}$$

In Handout 5, I talked about these maps in general, when $A, B, C, D \in \boldsymbol{C}$. Here I want to take $A, B, C, D \in \boldsymbol{R}$. We will call such a Mobius transformation a *real Mobius transformation*.

Exercise 6: Let M be a real Mobius transformation. Prove $M(\boldsymbol{H}^2) = \boldsymbol{H}^2$.

Exercise 7: Let b and c be two points in $\boldsymbol{H}^2$. Prove that

$$\Delta(M(b), M(c)) = \Delta(b, c).$$

In other words, and real Mobius transformation is an isometry of $\boldsymbol{H}^2$.

In Handout 5, I showed that Mobius transformations map circles to circles. Here is one additional piece of information about Mobius transformations.

Lemma 3.1 *A mobius transformation M preserves angles between curves.*

Proof: We can think of M as a map from $\boldsymbol{R}^2$ to $\boldsymbol{R}^2$, and write

$$M(x, y) = (M_1(x, y), M_2(x, y)).$$

We are interested in what happens at some point $p \in \boldsymbol{R}^2$ such that $M(p) \in \boldsymbol{R}^2$ as well. We can consider the matrix of first partial derivatives of M at p:

$$dM = \begin{bmatrix} \frac{\partial M_1}{\partial x} & \frac{\partial M_1}{\partial y} \\ \frac{\partial M_2}{\partial x} & \frac{\partial M_2}{\partial y} \end{bmatrix}.$$

On small scales, M behaves like this linear transformation. In particular, M preserves angles at p if and only if $dM|_p$ is a *similarity*–i.e. a rotation followed by a dilation. The linear transformation $dM|_p$ is a similarity if and only if it maps circles to circles. Since M maps circles to circles, so does $dM|_p$. Therefore, $dM|_p$ is a similarity. Therefore, M preserves angles at p. ♠

Now we know that every real Mobius transformation is a symmetry of the model: It maps geodesics to geodesics and preserves distances.

4 Hyperbolic Reflections

Complex conjugation $z \to \overline{z}$ is defined by the equation

$$\overline{x+iy} = x - iy. \tag{4}$$

The map $z \to -\overline{z}$ is just reflection in the vertical geodesic coming out of the point 0. This map is a hyperbolic isometry, and also fixes every point on the vertical geodesic.

As another example, the map Let $R(z) = 1/\overline{z}$ is a hyperbolic isometry that fixes every point on the geodesic connecting -1 to 1. Such maps are called *hyperbolic reflections*. That is, a hyperbolic reflection is an isometry that fixes every point on some geodesic.

Exercise 8: Let γ be any hyperbolic geodesic. Prove that there is a hyperbolic reflection that fixes every point on γ.

Exercise 9: Let p be any point in $\boldsymbol{H}^2$ and let θ be and angle. Prove that there is a hyperbolic isometry that fixes p and rotates by angle 2θ about p. (Hint: consider the product of two reflections in geodesics through p that meet at angle θ.)

Exercise 10: Let γ be any geodesic in $\boldsymbol{H}^2$ and let $d > 0$ be any number. Finally let $p \in \gamma$ be any point. Prove that there is a hyperbolic isometry T such that $T(\gamma) = \gamma$ and $\Delta(p, T(p)) = 2d$. In other words, T is a translation along γ by $2d$ units. (Hint: Consider the product of reflections that each fix different geodesics perpendicular to γ.)

Exercise 11: Prove that the product of two hyperbolic reflections is a Mobius transformation. You can do this just by calculation.

After having done these exercises, you will see that the symmetries of the hyperbolic plane include reflections, rotations, and translations, just like Euclidean symmetries. Of course, these symmetries look a little bit wierd to our Euclidean eyes.