A Result about Fiber Products:

The purpose of this note is to prove a result about the fiber product. I guess that the result here is well-known, but I don't know where to find it in the literature. the hypotheses of the result are somewhat artificially restricted for ease of exposition.

Let $\boldsymbol{R}/\boldsymbol{Z}$ be the circle. We call the map $f : \boldsymbol{R}/\boldsymbol{Z} \to \boldsymbol{R}/\boldsymbol{Z}$ a *nice map* if f has degree 1, is piecewise linear, and is not constant on any interval. A nice map is allowed to reverse direction. For instance, it might wind 999 times around in one direction and then 1000 times around in the other direction.

We call the places where f locally reverses direction the *fold points*. We call $t = f(s)$ a *fold value* if s is a fold point for f. We call two nice maps f_1 and f_2 *unrelated* if they have no common fold values.

Let $\boldsymbol{T} = (\boldsymbol{R}/\boldsymbol{Z})^2$. Given two nice maps f_1, f_2 we can form the *fiber product*

$$H(f_1, f_2) = \{(s_1, s_2) \in \boldsymbol{T} |\ f_1(s_1) = f_2(s_2)\}. \tag{1}$$

Here is the result.

Theorem 0.1 *Suppose f_1 and f_2 are unrelated nice maps. Then $H(f_1, f_2)$ is a polygonal 1-manifold which has exactly one connected component that is homologically nontrivial in $\boldsymbol{T}$. When suitably oriented, the one nontrivial component represents $(1, 1)$ in homology $H_1(\boldsymbol{T})$.*

It is not hard to imagine or prove that $H(f_1, f_2)$ is a manifold that represents $(1, 1)$ in homology. The interesting part of the result is that there is *exactly* one essential component of $H(f_1, f_2)$.

We will prove Theorem 0.1 through a series of 4 lemmas.

Lemma 0.2 *H is a polygonal 1-manifold.*

Proof: Given two partitions $\{I_i\}$ and $\{J_j\}$ of $\boldsymbol{R}/\boldsymbol{Z}$ into intervals, we can take the product and get a partition of $\boldsymbol{T}$ into rectangles $\{R_{ij}\}$ with $R_{ij} = I_i \times J_j$. Since f_1 and f_2 are unrelated, we can choose these partitions so that the restriction of each function to each interval is linear and injective, and no vertex of an R_{ij} belongs to H. The locations of the fold points force us to choose certain breaks in the partitions, but otherwise we choose the breaks generically.

Let $H_{ij} = H \cap R_{ij}$. By construction H_{ij} is either the emptyset or a line segment which connects the interior point of some edge of R_{ij} to the interior point of some other edge of R_{ij}. Consider the picture around an endpoint p of H_{ij}. Let R' be the rectangle adjacent to R_{ij} across the edge containing p. Since $H \cap R'$ is not the emptyset, $H \cap R'$ has the structure just mentioned. In particular, H_{ij} meets a unique line segment of H at p. This shows that H is a polygonal 1-manifold. ♠

Lemma 0.3 *H has an orientation with the following properties:*

- *Whenever the generic vertical geodesic $x = x_0$ intersects H at a point (x_0, y), the relevant segment points to the right if and only if $f_2'(y) > 0$.*
- *Whenever the generic horizontal geodesic $y = y_0$ intersects H at a point (x, y_0), the relevant segment points to the top if and only if $f_1'(x) > 0$.*

Proof: Here is the construction. If $f_1(I_j)$ and $f_2(J_j)$ are not disjoint, then they overlap in one of 4 possible ways. At the same time, there are 4 possible orientations for these segments, depending on the signs of the derivatives f_1' and f_2'. All in all, there are 16 different possibilities. For each of these possibilities, we choose an orientation for the corresponding segment of H, according to the scheme shown in Figure 1.

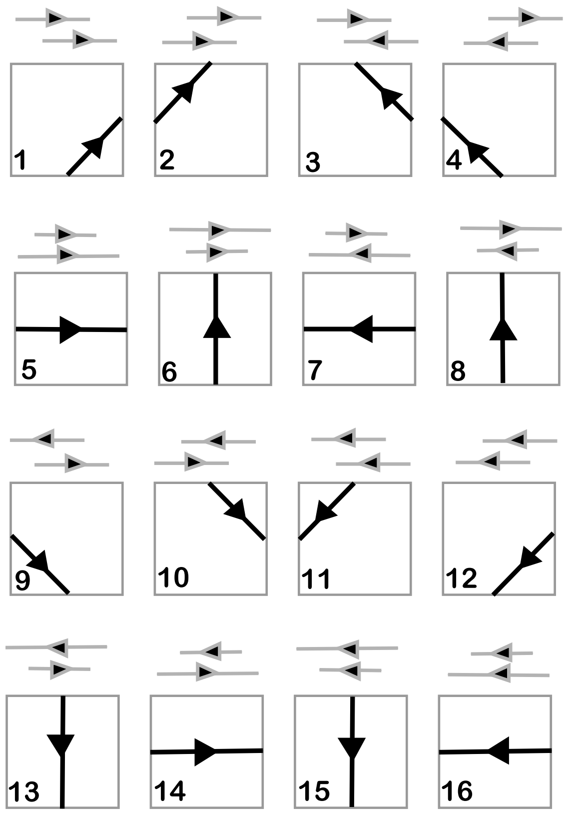

Figure 1: The orientation on the fiber product

A case-by-case check shows that this scheme defines a consistent orientation on H. Figure 2 shows how Cases 1,2 fit together and how Cases 1,3 fit together.

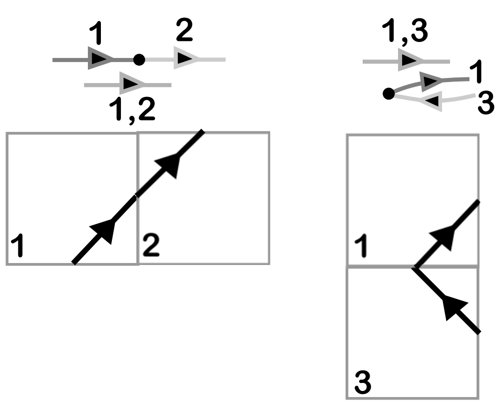

Figure 2: Adjacent pairs of segments and their tiles.

Now we check how the geodesic $x = x_0$. intersects one of our tiles. In all cases, the arrow in Figure 1 points to the right if and only if the lower of the two segments (corresponding to $f_2(J_j)$ points to the right. Similarly, we check how the geodesic $y = y_0$ intersects our tiles. In all cases, the arrow in Figure 1 points up if and only if the upper of the two segments (corresponding to $f_1(I_i)$) points to the right. ♠

From now on we equip H with the orientation given above, and we call it the *natural orientation*. Since H is oriented, it makes sense to ask which homology class H represents in $H_1(\boldsymbol{T})$.

Lemma 0.4 *H represents the element $(1,1)$ in $H_1(\boldsymbol{T})$.*

Proof: If suffices to show that the geodesics $x = x_0$ amd $y = y_0$ each intersect H once, counting the orientations. Consider $x = x_0$. Each intersection point with this geodesic corresponds to a parameter value y where $f_2(y) = f(x_0)$. The orientation points to the right if and only if $f_2'(y) > 0$. But the number of times $f_2'(y) > 0$ is one more than the number of times $f_2'(y) < 0$ because f_2 has degree 1. In other words, $f_2(\boldsymbol{R}/\boldsymbol{Z})$ crosses a point righwards one more time than it crosses leftwards. This proves our claim for the geodesic $x = x_0$. A similar argument works for the geodesic $y = y_0$. ♠

Now we know that H represents $(1,1)$ in $H_1(\boldsymbol{T})$. Two distinct and nontrivial homology classes in $\boldsymbol{T}$ intersect unless they represent the same homology classes or their sum is 0 in homology. Since H is an embedded 1-manifold, all the homologically nontrivial components of H represent either $(1,1)$ or $(-1,-1)$. Moreover, the number of $(1,1)$ representatives is one more than the number of $(-1,-1)$ representatives. The last step finishes the proof.

Lemma 0.5 *An arbitrary non-trivial component h of H represents $(1,1)$ in homology.*

We can find a piecewise linear map $a : \boldsymbol{R}/\boldsymbol{Z} \to \boldsymbol{T}$ such that $a = (a_1, a_2)$ parametrizes h, and each a_j is a degree 1 map. Define $b = f_j \circ a_1$. This map is independent of j and has degree 1.

The parametrization a gives h a second orientation of h which we call the *forced orientation*. The component h represents the element $(1,1)$ in $H_1(\boldsymbol{T})$

with respect to the forced orientation. So, to finish the proof, we need to show that the forced orientation and the natural orientation coincide.

Given $t \in \boldsymbol{R}/\boldsymbol{Z}$ we can compare the signs of $f_2'(a_2(t))$ and $a_1'(t)$. The former quantity determines the direction that h points across the vertical line through $a(t)$. The latter quantity determines the direction that h points across the vertical line through $a(t)$. The two orientations agree iff the two quantities have the same sign. By the Chain Rule, $f_2'(a_2(t))$ is positive if and only if $a_2'(t)$ and $b'(t)$ have the same sign. Therefore the two orientations agree if there is any point t such that

$$a_1'(t)a_2'(t)b'(t) > 0 \tag{2}$$

Note that $a_j(s) = a_j(t)$ implies that $b(s) = b(t)$. This is because $b = f_j \circ a_j$.

We will suppose that Equation 2 fails for all t and we will derive a contradiction. We can find lifts $A_1, A_2, B : \boldsymbol{R} \to \boldsymbol{R}$ of a_1, a_2, b respectively. Each function F is such that $F(x+1) = F(x)+1$. The lifted functions also satisfy the same property as above: If $A_j(s) = A_j(t)$ then $B(s) = B(t)$. Moreover $A_j' = a_j'$ and $B' = b'$. So, $A_1'(t)A_2'(t)B'(t) < 0$ whenever all these derivatives are defined. In particular, these derivatives cannot all be positive.

Say that a point $t \in \boldsymbol{R}$ is a *peak* if the function $B(t) - t$ has a global maximum at t. A peak exists because the function $B(t) - t$ is periodic. Let t_0 be a peak. By construction, $B(s) < B(t_0)$ for all $s < t_0$. For $\epsilon > 0$ sufficiently small, we have $B'(t_0 - \epsilon) \geq 1 > 0$. We pick ϵ so small that no derivative changes sign on $[t_0 - \epsilon, t_0]$. Since not all derivatives are positive, have $A_j'(t_0 - \epsilon) < 0$ for some j. By the Fundamental Theorem of Calculus, $A_j(t_0 - \epsilon) > A_j(t_0)$. Since $A_j(t_0 - 1) < A_j(t_0)$ there is some $s \in (t_0 - 1, t_0)$ such that $A_j(s) = A_j(t_0)$. But then $B(s) = B(t_0)$. This is a contradiction. ♠