Pan Galactic Division

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1 Introduction

The purpose of these notes is to explain Peter Doyle and Cecil Qiu's proof [2] that "division by N" is possible on a set-theoretic level. To state the result formally, let $S \times N$ stand for $S \times \{0, ..., (N-1)\}$. Here S is an arbitrary set.

Theorem 1.1. *Let A and B be sets. If there is an injective map from $A \times N$ to $B \times N$ then there is an injective map from A to B.*

The significance of Doyle and Qiu's proof is that it produces an effective (i.e. canonical) injection from $A \to B$ in a simple and straightforward way. In particular, their proof avoids using the Well-Ordering Principle or any other form of the Axiom of Choice. For comparison, in 1949 Tarski [3] gave an effective proof, but Tarski's proof is complicated. Conway and Doyle [1] sought a simpler proof, but found nothing as simple as the proof here. For more on the long and tangled history of this problem, see [2].

Theorem 1.1 is an immediate consequence of induction and the following result.

Theorem 1.2. *Let N be any positive integer. If there is an injection from $A \times N$ to $A \times N$ then there is an injection from $A \times (N-1)$ to $B \times (N-1)$.*

The proof is the same regardless of the choice of N. Following Doyle and Qiu, I'll give the proof when $N = 4$, so that it looks like a game of cards. Doyle and Qiu call their game Pan Galactic Division, because they contend that a definite fraction of intelligent civilizations in the universe will have hit upon their canonical algorithm.

2   The Rules of the Game

Think of $B \times 4$ as a deck of cards, where the pictures on the cards are the members of B and the suits **spades, hearts, diamonds, clubs** are just other names for $3, 2, 1, 0$. Think of A as a set of players, each staring at 4 spots in front of them, named **spades, hearts, diamonds, clubs** in order from left to right.

Enough of the cards in the deck are dealt out to the players so that each player has a hand of 4 cards, placed into the 4 spots. (The suit on a card need not match the suit which names the spot it is in.) The injective map from $A \times 4$ to $B \times 4$ just maps each pair (player, spot) to the card in that location. The game has alternating rounds, called *Shape Up* and *Ship Out*.

(1) Shape Up: Each player having at least one spade arranges to have a spade as the leftmost card. This is done by swapping the leftmost spade with the leftmost card, if such a swap is necessary. Figures 1 and 2 show an example, before and after the player shapes up.
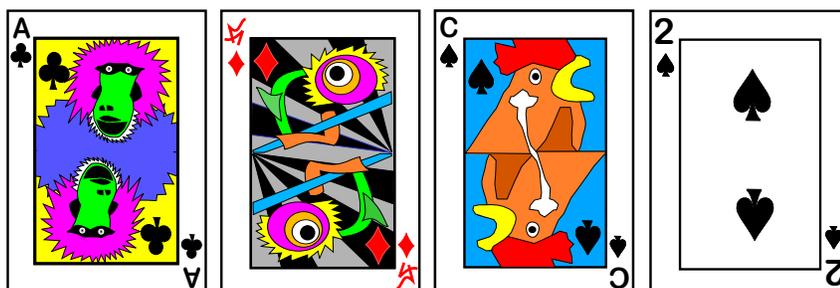
Figure 1: Before shaping up: The Chicken of spades is the leftmost spade.
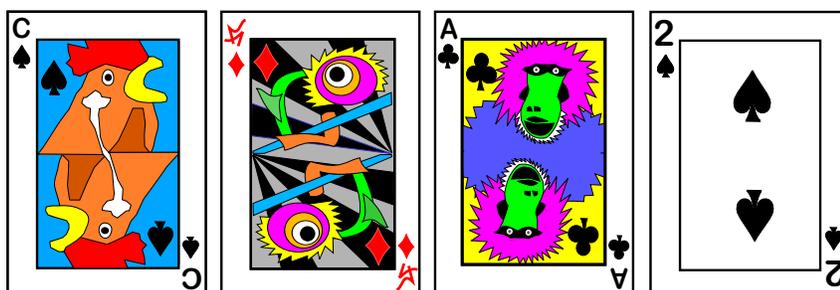
Figure 2: After shaping up: The Chicken of spades swaps with Ape of clubs.

Names of the Hands: All players shape up simultaneously. Once that is done, any hand which has a spade in it will also have a spade in the leftmost spot. In this case, the name of this leftmost spade names the hand. In our example, the name of the hand is "Chicken", because the Chicken of spades occupies the leftmost position. (Since the sets A and B might be very large, we can't expect our cards to be as in a traditional deck.) The hands without spades are not named.

(2) Ship Out: After a player shapes up, any named hand in her hand will have a spade on the left. Say that a *bad spade* is a spade in a named hand which is not the leftmost spade. In our example above, the 2 of spades is a bad spade. A hand can have up to three bad spades.

In the Ship Out round, each player with a bad spade swaps their leftmost bad spade with another card in the game, according to the following rule. Recall that the spots in front of the player are named spades, hearts, diamonds, clubs, from left to right. The player asks for the card which has the same suit as the name of the spot occupied by the leftmost bad spade, and the same name as the name of the hand. There is never any conflict: two players will not ask for the same card.

Let's continue with our example above. In our example, there is one bad spade, namely the 2 of spades. Even though this card lies in the rightmost spot, it is the leftmost bad spade, because it is the only bad spade. The 2 of spades occupies the clubs spot, according to our scheme, and it lies in the Chicken hand. Therefore, the player swaps the 2 of spades with the Chicken of clubs.

The Chicken of clubs either is part of another player's hand, or elsewhere in the player's own hand, or else is still in the deck; the distinction does not matter. Figure 3 shows our hand after the player ships out.
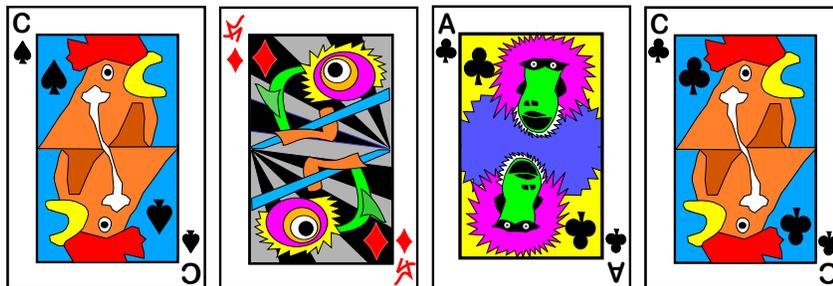
Figure 3: Two of spades is traded for the Chicken of clubs.

3 Playing the Game

The game is now played indefinitely, with the rounds alternating: shape up, ship out, shape up, ship out, ... Notice that a hand never loses its leftmost spade card, because the Ship Out rule will never require anyone to call for a spade. So, any player shapes up at most once.

The quality of a named hand is assessed according to how many cards have names which match the name of the hand and suits which match the name of the spot they occupy. Let's consider our example. Before shipping out, our player has 1-of-a-kind, so to speak, because the Chicken of spades is her only card which has the same name as the hand (by definition) and is in the correct position. After shipping out, our player has acquired the Chicken of clubs and placed it into the clubs position. Thus, now she has 2-of-a-kind. Notice that our player has improved her hand by shipping out.

As our example suggests, a player always improves her hand by shipping out a spade. A player's hand can have a spade *shipped in* by having one of her cards called away. For instance, in the example below, our player loses the ape of clubs and acquires the bolt of spades. The bolt of spade is a bad spade in the diamonds position, as shown in Figure 4.
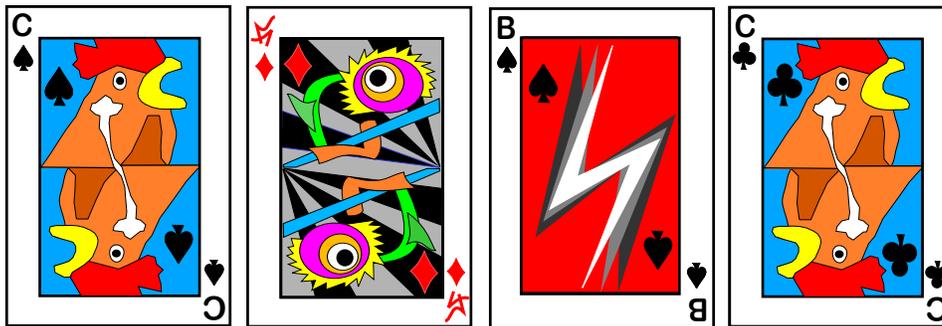
Figure 4: The bolt of spades occupies the diamonds position.

In the next round, our player ships out the bolt of spades and gets the Chicken of diamonds, as shown in Figure 5. In this way, she has improved her hand to 3-of-a-kind. So, any time a player participates in a Shipping Out round, either actively or passively, her hand improves after at most two more rounds of the game.
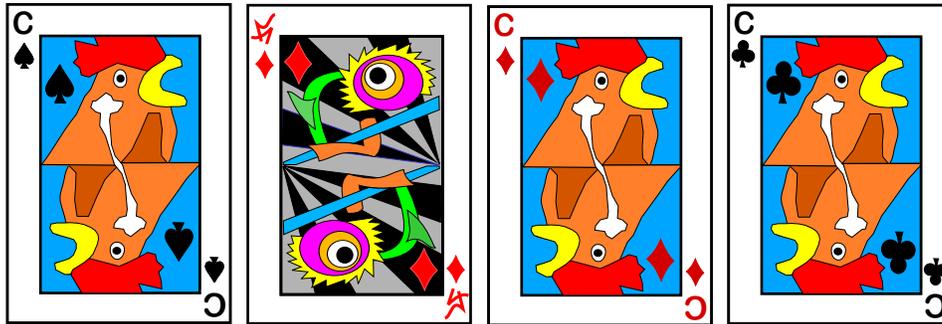
Figure 5: The bolt of spades is traded for the Chicken of diamonds.

4 Conclusion

From the discussion above, we conclude that during this indefinite game, a player's hand can change at most 8 times. Moreover, once a player's hand has stopped changing, the player has no bad spades. In other words, once the hand has stabilized, it has no spades in the 3 non-left spots.

We imagine that the game proceeds indefinitely. There is a well-defined non-spade card associated to each pair (player, non-left spot): We simply wait until the player's hand stabilizes - and it will stabilize - and then we see what card is occupying the relevant spot. This gives us an injective map from $A \times 3$ to $B \times 3$.

References

[1] Peter G. Doyle and John Horton Conway. Division by three, 1994, arXiv:math/0605779 [math.LO]. `http://arxiv.org/abs/math/0605779`.

[2] Peter G. Doyle and Cecil Qiu. Division by four, 2015, arXiv:1504.01402 [math.LO]. `http://arxiv.org/abs/1504.01402`.

[3] A. Tarski. Cancellation laws in the arithmetic of cardinals. *Fundamenta Mathematica*, 36:77–92, 1949.