Notes on Link Universality. Rich Schwartz: In his 1991 paper, *Ramsey Theorems for Knots, Links, and Spatial Graphs*, Seiya Negami proved a beautiful theorem about linearly embedded complete graphs. These notes give a more straightforward proof. Up to the last step, all the ideas essentially come from Negami's paper. For the last step, "putting a link on the twisted cubic" I couldn't understand Negami's proof so I found my own. Negami also proves his Theorem for embedded graphs, but I ignore this case for ease of exposition.

The Result: Let L be any link. Then there is an integer $R = R_L$ with the following property: Let S be any collection of R points in general position in $\boldsymbol{R}^3$. Then there is a union of polygons L', with vertices in S, having the same link type as L. An equivalent formulation is that any straight line embedding of a complete graph of size R_L has a union of cycles with link type L.

Warmup: Negami's Theorem is a close cousin of the following result from planar geometry. Given an integer n there is some other integer $N = N_n$ such that any N points in the plane in general position contains a convex n-gon. Here is a proof of that result. We order the points, from 1 to N. Let S_3 denote the set of ordered triples of $\{1, ..., N\}$. That is, an element of S_3 has the form (i, j, k) with $i < j < k$. We color an element (i, j, k) white if the corresponding triangle is positively oriented, and black in the other case. Ramsey's Theorem says that, once N is large enough, there is a subset $S' \subset S$, having size n, such that every element of S'_3 has the same color. But then the vertices of S' make a convex n-gon.

Corollary: A subset $S \subset \boldsymbol{R}^3$ is *clean* if S projects onto a finite set of points contained in the graph of a convex function. The points of the projection are the vertices of a convex polygon, and they can be ordered left to right. The warm-up result has the following corollary: Given any integer n, there is some integer N with the following property. If S is a subset of N general position points in $\boldsymbol{R}^3$, then (after rotating S if necessary) some n-element subset $S' \subset S$ is clean.

Proof: taking N large, we obtain a $2n$-element subset S' whose projection is the vertices of a convex polygon. We can divide this polygon into 2 halves, each of which is rotationally equivalent to a finite set of points on the graph of a convex function. One of the halves has at least n points.

Positivity: We call a linearly embedded complete graph clean if its vertices form a clean set. In view of the corollary, it suffices to prove Negami's Theorem for clean complete graphs. Let Γ be a clean complete graph. We orient the edges of Γ so that they point from left to right, when projected into the plane. In other words, the tail vertex of each edge has smaller x-coordinate than the head vertex.

We call a pair of crossing edges in the graph *positive* if they cross as in Figure 1, with the edge having tail with larger x coordinate going over the other one.

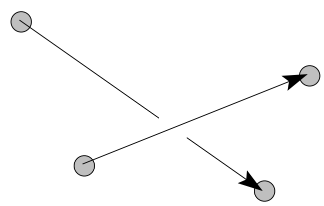

Figure 1: A positive crossing

We call Γ *positive* if every pair of crossing edges is positive. Now we prove the following result. Given any n there is some N with the following property: If Γ is a clean complete graph of size N, then (up to mirrror reflections) Γ contains a positive clean complete graph Γ' of size n.

The proof is just like the warmup. S be a clean set of N points in $\boldsymbol{R}^3$. Let $S_4(S)$ denote the set of ordered 4-tuples of $\{1, ..., N\}$. Given an element (i, j, k, l), we consider the corresponding 4-tuple of points. There is a unique way to make two crossing edges from these points. We color (i, j, k, l) white if these edges form a positive crossing, and black otherwise. Ramsey's Theorem gives us a set $S' \subset S$ of size n such that all elements of $S_4(S')$ get the same color. If the color is white, we are done. If the color is black, we reflect the picture in the xy plane.

The End of the Proof: Below we will prove the following claim. Given any link L, there is some $N = N_L$ such that any positive clean complete graph of size N contains a cycle with link type L. Call this claim $C(+)$. At the same time, one can define negative clean complete graphs and make a similar claim. Call this claim $C(-)$. Since $C(+)$ is supposed to hold for all links, including the mirror image of a given link, $C(+)$ for all links implies $C(-)$ for all links.

From what we have proved above, every sufficiently large linearly embedded complete graph contains either a positive or a negative clean complete graph of size N_L. Hence, claims $C(+)$ and $C(-)$ finish the proof. However, since $C(+)$ implies $C(-)$, just $C(+)$ alone finishes the proof. So, to finish the proof of Negami's Theorem, we just have to prove $C(+)$.

The Twisted Cubic Let K_n denote the complete graph on n vertices. Any two clean positive embeddings of K_n are equivalent in the following sense: They contain precisely the same links: The obvious bijection between two such embeddings is such that the corresponding cycles in each one have the same planar diagrams. Given this equivalence, it is useful to have a nice model for a clean positive complete graph. The twisted cubic provides such a model. The twisted cubic is the curve X with parametric equations (t, t^2, t^3). This curve projects to the parabola (t, t^2). Any collection of n points on X gives rise to a clean positive embedding of K_n. So, for the purposes of proving $C(+)$, we just have to show that any link can be realized as a polygon having vertices on the twisted cubic.

Bridge Position: Let $f(x, y, z) = x$ denote the map which takes the first coordinate. Say that a smooth link L is in *bridge position* if if $f(L) = [0, 1]$ and the only critical points of f are either global minima, namely $f^{-1}(0)$, and global maxima, namely $f^{-1}(1)$. When L is in bridge position, L is realized as a bipartite graph where the (not necessarily straight) edges connect minima to maxima. It is a well-known result that every link can be put in bridge position. Intuitively, you just clasp all the minima of the link with your left hand, and all the maxima with your right hand – then you pull the link tight, like a rubber band. So, as a first step to proving $C(+)$, we put the given link in bridge position.

Positive Bridge Position: Let L be a link in bridge position. Thinking of L as a bipartite graph, we orient each of the strands of L from left to right. We say that L is in *positive bridge position* if all the crossings are positive. Every link can be placed in positive bridge position: Scanning the link from left to right, you look for the first negative crossing. Assuming you have found a negative crossing, you give the right half of the link a twist while keeping (most of) the left half fixed. This twist has the effect of removing the negative crossing at the expense of adding some new positive crossings. Just do this finitely many times to eliminate all negative crossings.

Realizing the Link (This part of the notes diverges significantly from what Negami does.) Given a link in positive bridge position, we can spread out the crossings so that they appear sequentially as in Figure 2. The portion between each set of vertical lines is one of a small number of standard types. We will call such a portion a *segment*. (The vertical lines are not part of the link.) The crossings are all understood to be positive. The link below has 6 segments.

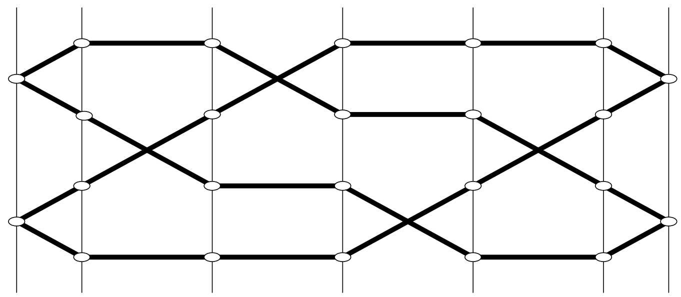

Figure 2: A link in positive bridge position

Each individual segment of the link L can certainly be realized as a collection of arcs with endpoints on the twisted cubic. See Figure 4 below. Moreover, we can realize each segment individually in such a way that it lies in a thin tubular neighborhood of a single segment. Then we can concatenate the individual realizations. This simple procedure almost works. Unfortunately, it produces a link L^* which is possibly different from L. However, let L' denote the new link in which every other segment has been reversed top-to-bottom. Figure 3 shows what we mean by way of example. The red colored segments have been reversed, but only segment 4 notices the difference. Segments 2 and 6 are symmetric.

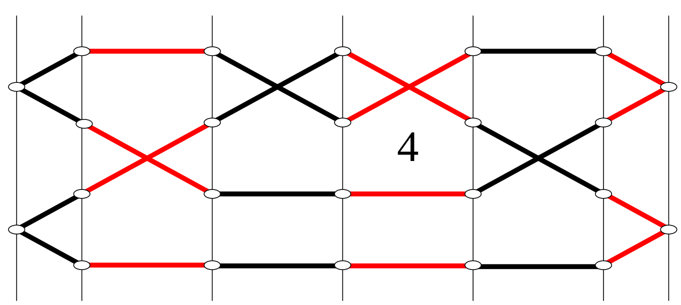

Figure 3: The twist of the link from Figure 2.

The lucky thing is that $(L')^*$ is isotopic to L. Figure 4 shows this in action. Figure 4 shows the first 4 segments of L' embedded on the twisted cubic. (We have drawn the projection of the twisted cubic as an arc of a circle, to get a better picture.) One goes from $(L')^*$ to L simply by untwisting the picture, starting at the right and moving left, making a (roughly) 180 degree twist at each juncture. The positivity makes this twisting possible. For instance, all the red lines of segment 4 go over the black lines of segment 3, allowing one to straighten out segment 4 with a twist.

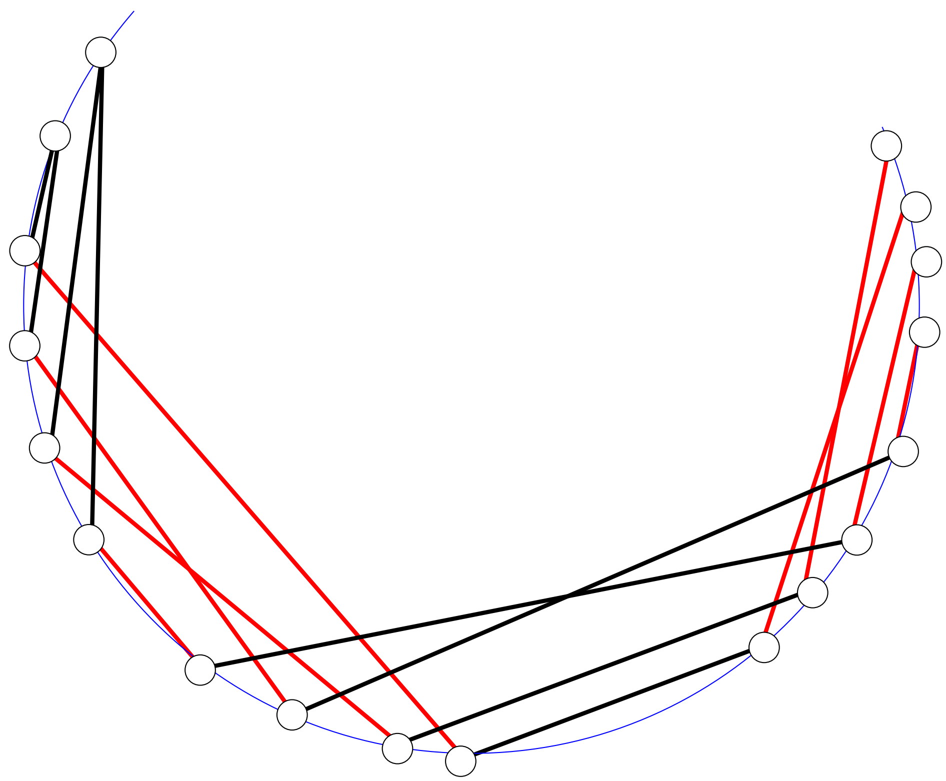

Figure 4: Realizing the first 4 segments of L'.