Outer Billiards, Quarter Turn Compositions, and Polytope Exchange Transformations

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Abstract

In this paper we define and study certain piecewise affine maps of the infinite strip, which we call quarter turn compositions. These maps capture essentially all the information contained in polygonal outer billiards and have higher dimensional compactifications which are polytope exchange transformations (PETs). The PETs which arise are based on pairs of incommensurable lattices in Euclidean space, and we call them double lattice PETs. This paper represents a step towards a general theory of polygonal outer billiards, and also produces multi-parameter families of double lattice PETS in every dimension.

1 Introduction

1.1 Overview

Interval exchange transformations (IETs) come up frequetly in dynamics. These maps arise in billiards, surface foliations, and Teichmuller Theory. There is a vast literature on IETs. See, for instance, [**K**], **M**], [**R**] [**V**], [**Z**]. Considerably less is known about higher dimensional analogs, known as polytope exchange transformations (PETs).

To define a PET, one starts with a polytope (or a flat manifold) M that has been partitioned into "the same" smaller polytopes in two different ways, say $M = \sqcup_{i=1}^{n} P_i$ and $M = \sqcup_{i=1}^{n} Q_i$. Here there is a translation f_i such that

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$f_i(P_i) = Q_i$. This structure allows us to define $f : M \to M$ so that f agrees with f_i on the interior P_i. The map f is the PET.

In this paper we will study certain piecewise affine maps of the infinite strip

$$\mathbf{S} = \boldsymbol{R} \times [-1/2, 1/2], \tag{1}$$

which we call *quarter turn compositions* (QTCs). These QTCs capture essentially all the information contained in polygonal outer billiards (Theorem 1.1) and at the same time have higher dimensional compactifications (Theorem 1.2) with fairly explicit descriptions (Theorem 1.3) In general, these compactifications are affine [1] PETs, but in all cases coming from outer billiards, and in some other cases as well, the square of the affine PET is an ordinary PET which we call a *double lattice PETS*.

The work in [**S2**], which just analyzes some 3 dimensional PETs associated to outer billiards on kite shaped quadrilaterals, suggests that the double lattice PETs in general have very rich dynamical properties. The work in [**S3**], and the somewhat related work in [**Hoo**], suggests that these PETs may even have a renormalization theory.

1.2 Polygonal Outer Billiards

To define a polygonal outer billiards system, we start with a convex polygon P. Given a point $x_0 \in \boldsymbol{R}^2 - P$, one defines x_1 to be the point such that the segment $\overline{x_0x_1}$ is tangent to P at its midpoint and P lies to the right of the ray $\overrightarrow{x_0x_1}$. The iteration $x_0 \to x_1 \to x_2 ...$ is called the *forwards outer billiards orbit* of x_0. It is defined for almost every point of $\boldsymbol{R}^2 - P$. The backwards orbit is defined similarly.

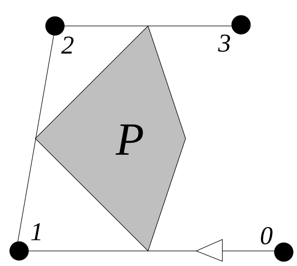

Figure 1.1: outer billiards relative to P.

[1] An affine PET has the same definition as an ordinary PET, except that the maps f_i are all affine maps, and the linear part does not depend on i.

B.H. Neumann [**N**] introduced outer billiards in the late 1950s, and J. Moser [**M1**] popularized the system as a toy model for celestial mechanics. See [**T1**], [**T2**], and [**DT**] for expositions of outer billiards and many references. One of the central questions in the subject has been the *Moser-Neumann question* [**M2**]: Does there exist a P for which there are unbounded orbits?

One basic result on this topic is the one due (independently) to Vivaldi-Shaidenko [**VS**], Kolodziej [**K**], and Gutkin-Simanyi [**GS**]. This result states that outer billiards with respect to P has only bounded orbits if P is *quasi-rational*. We will explain what this means below, but we mention here that both regular polygons and polygons with rational vertices are quasi-rational.

In [**S1**] we proved that outer billiards has unbounded orbits relative to a certain convex quadrilateral called the *Penrose kite*, and in [**S2**] we showed that outer billiards on a kite has unbounded orbits if and only if the kite is irrational – meaning that one of the diagonals of the kite divides it into triangles having irrationally related areas. So, far, these are the only [2] (un)boundedness results known about polygonal outer billiards.

All our papers on polygonal outer billiards have followed a 3-step pattern:

1. Relate a first return map of outer billiards to the so-called Pinwheel map. Essentially, the Pinwheel map is the same as a QTC, but we have not presented it this way previously.

2. Show that the Pinwheel map has a higher dimensional compactification which is a PET.

3. Prove dynamical theorems about outer billiards by studying the properties of the PET – either its symbolic dynamics [**S1**], its Diophantine properties [**S2**], or a renormalization scheme [**S3**].

Since writing [**S2**] we have wanted to work out the theory of polygonal outer billiards in something like full generality. One could say that our paper [**S4**] carries out "step 1" of the three step outline above, for general polygons. The purpose of this paper is to carry out "step 2".

[2] We also mention that Dolgopyat and Fayad [**DF**] proved that outer billiards has unbounded orbits relative to the half-disk and closely related shapes. Their work, however, does not really fit into the present context.

1.3 Quarter Turn Compositions

Let $\square$ be a rectangle with sides parallel to the coordinate axes. We define a *quarter turn* of $\square$ to be the order 4 affine automorphism of $\square$ which maps the right edge of R to the bottom edge of $\square$. This map essentially twirls $\square$ one quarter of a turn clockwise. For any $a > 0$ we distinguish 2 tilings of the strip $\mathbf{S}$ by $a \times 1$ rectangles. In *Tiling 0*, the origin is the center of a rectangle. In *Tiling 1*, the origin is the center of a vertical edge of a rectangle. For $q = 0, 1$ let $R_{q,a}$ denote the may which gives a quarter turn to each rectangle in Tiling q. The map $R_{q,a}$ is a piecewise affine automorphism of $\mathcal{S}$, defined everywhere except the vertical edges of the rectangles. We call $R_{q,a}$ a *quarter turn*.

We define the *shear*

$$S_s = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} \tag{2}$$

Here $s > 0$. The map S_s is a shear of $\mathbf{S}$ which fixes the centerline pointwise, moves points with positive y-coordinate backwards and points with negative y-coordinate forwards.

We define a *quarter turn composition* (QTC) to be a finite alternating composition $\mathcal{T}$ of quarter turns and shears. That is,

$$\mathcal{T} = S_{s_n} \circ R_{q_n,r_n} \circ \cdots \circ S_{s_1} \circ R_{q_1,r_1}. \tag{3}$$

- $q_1, ..., q_n \in \{0, 1\}$ specify the tiling offsets.
- $r_1, ..., r_n$ are the parameters for the widths of the rectangles.
- $s_1, ..., s_n > 0$ are the parameters for the shears.

We call n the *length* of the QTC.

It is convenient to define

$$\alpha_i = r_n/r_i \tag{4}$$

The choice of n as a special index is arbitrary; any other choice leads to the same definitions. We call $\mathcal{T}$ *quasi-rational* if $\alpha_i \in \boldsymbol{Q}$ for all i.

We call $\mathcal{T}$ *finitary* if $\mathcal{T}$ is a piecewise translation, and the set

$$\{\mathcal{T}(p) - p |\ p \in \mathbf{S}\} \tag{5}$$

of possible translations is finite.

The reflection ρ through the origin commutes with $\mathcal{T}$. We declare the orbits O and $\rho(O)$ of $\mathcal{T}$ to be equivalent. Given a convex polygon P, let $U(P)$ denote the set of unbounded outer billiards orbits relative to P. Let $U(\mathcal{T})$ denote the (possibly empty) set of equivalence classes unbounded orbits of $\mathcal{T}$.

Theorem 1.1 *Let P be a convex n-gon that has no parallel sides. There exists a quarter turn composition $\mathcal{T}_P$, of length n, together with a canonical bijection between $U(P)$ and $U(\mathcal{T}_P)$. The map $\mathcal{T}_P^2$ is finitary.*

Remarks:
(i) Theorem 1.1 is really just a reinterpretation of the [**S4**, Pinwheel Theorem]. All the hard work was done there.
(ii) From our construction given in §3, it is immediate that $\mathcal{T}_P$ is quasi-rational if and only if P is quasi-rational. One can take this as the definition of what it means for a convex polygon to be quasi-rational.
(iii) In §6 we will give an example of a length 3 rational QTC $\mathcal{T}$ such that $\mathcal{T}$ is rational and finitary but also has unbounded orbits. Our example, which cannot come from Theorem 1.1, shows that the QTCs which come from Theorem 1.1 are somewhat special.
(iv) Probably there is a similar result for polygons having some parallel sides, but this case presents some tedious complications we prefer to avoid.

1.4 The Compactification Theorem

Define the unit torus

$$\widehat{\mathbf{S}} = \boldsymbol{R}^{n+1}/\boldsymbol{Z}^{n+1}. \tag{6}$$

Theorem 1.2 *Suppose that $\mathcal{T}$ is a length-n quarter turn composition. Then there is a locally affine map $\Psi : \mathbf{S} \to \widehat{\mathbf{S}}$ and an affine PET, $\widehat{\mathcal{T}} : \widehat{\mathbf{S}} \to \widehat{\mathbf{S}}$, such that $\Psi \circ \mathcal{T} = \widehat{\mathcal{T}} \circ \Psi$.*

- *The map Ψ is injective if and only if $\mathcal{T}$ is not quasi-rational.*
- *The closure of $\Psi(\mathbf{S})$ is a sub-torus of dimension $1 + d$, where d is the $\boldsymbol{Q}$-rank of $\boldsymbol{Q}(\alpha_1, ..., \alpha_{n-1})$.*
- *If $\mathcal{T}^k$ is finitary, then the restriction of $(\widehat{\mathcal{T}})^k$ to the closure of $\Psi(\mathbf{S})$ is an ordinary PET.*

Remarks:
(i) In case $\mathcal{T}$ is as in Theorem 1.1, the map $\widetilde{\mathcal{T}}^2$ is a PET. Thus, Theorem 1.2 provides a PET compactification for what is essentially the same thing as the outer billiards dynamics in every case.
(ii) The action of $\widehat{\mathcal{T}}$ on all of $\widehat{\mathbf{S}}$ is never an ordinary PET. This is not a contradiction: In case $\mathcal{T}$ is finitary, the image $\Psi(\mathbf{S})$ is not dense. On the other hand, there are many examples where $\widehat{\mathcal{T}}^2$ acts as a PET on all of $\widehat{\mathbf{S}}$.

One can specify an affine PET by giving a triple (X_1, X_2, I), where X_1 and X_2 are polyhedral fundamental domains for $\boldsymbol{Z}^{n+1}$ and

$$I : X_1 \to X_2 \tag{7}$$

is a linear isomorphism. The affine PET is given by

$$[X_1, X_2, I] := \Pi_2 \circ I \circ \Pi_1^{-1} \tag{8}$$

Here Π_j is the canonical map from X_j to $\widehat{\mathbf{S}}$. The map $\Pi_j^{-1} : \widehat{\mathbf{S}} \to X_j$ is defined as follows: Lift to $\boldsymbol{R}^{n+1}$ then translate by the appropriate integer vector. The map $[X_1, X_2, I]$ is defined and locally affine on $\Pi_1(X_1)$, but need not extend to all of $\widehat{\mathbf{S}}$. When I is an involution, $[X_1, X_2, I]^2$ is an ordinary PET.

Theorem 1.3 *The affine PET from Theorem 1.2 is conjugate to a map* $[X_1, X_2, I]$*, where* X_1 *and* X_2 *are parallelotope fundamental domains for* $\boldsymbol{Z}^{n+1}$*, centered at the origin. The map* I *fixes pointwise a codimension* 2 *subspace of* $\boldsymbol{R}^{n+1}$ *and preserves each* 2*-plane parallel to* $\Psi(\mathbf{S})$*.*

We call $[X_1, X_2, I]$ *standard* if I has eigenvalues $-1, -1, 1, ..., 1$. Whenever the QTC is comes from an outer billiards system, the compactification is standard. In the standard case, we can express the map $[X_1, X_2, I]$ in a more symmetric way. This leads to a PET which we call a *double lattice PET*. See §4.5 for the definition and a discussion.

We also attach to each orbit of $[X_1, X_2, I]^2$ a lattice path in $\boldsymbol{Z}^{n+1}$ which we call the *arithmetic graph*. See §5.1. Assuming that the orbit $\widehat{O} = \Psi(O)$ is the image of an orbit of the QTC, the arithmetic graph of $\widehat{O}$ has an affine projection into $\mathbf{S}$ which coincides with O. Thus, one can recover the dymamics of the QTC from the symbolic dynamics on the compactification. Previously we had just defined the arithmetic graph for outer billiards on kites, and in that setting we found it an extremely useful tool. We have not tried to study the arithmetic graph in general, however.

1.5 Overview

In §2 we prove Theorem 1.1. in §3 we prove Theorem 1.2. In §4 we prove Theorem 1.3. In §5 we introduce the arithmetic graph and prove all the claims about it that we mentioned above. In §6 we give some concrete examples of the objects discussed above.

1.6 Acknowledgements

This work benefitted greatly from conversations with Eugene Gutkin and Sergei Tabachnikov about polygonal outer billiards, and with Pat Hooper, Rick Kenyon, and John Smillie about PETs and related matters.

In particular, John Smillie has an alternate way to prove a statement close to Theorem 1.2. Smillie's approach is more conceptual but less explicit, and it applies not to a QTC but to the pinwheel map. Smillie's idea is to embed this system into an infinite dimensional vector space using a tensor product construction akin to Dehn's invariant, and then to observe that the image is finite dimensional. While this idea does not directly arise in the constructions here, it somehow had its influence.

2 Connection to Outer Billiards

2.1 Strip Maps

We introduce a construction that is intermediate between outer billiards and quarter-turn systems. Let Σ be an infinite strip in the plane and let V be a vector that spans Σ in the sense that the tail of V lies on one component of $\partial\Sigma$ and the head of V lies on the other component. See Figure 2.1.

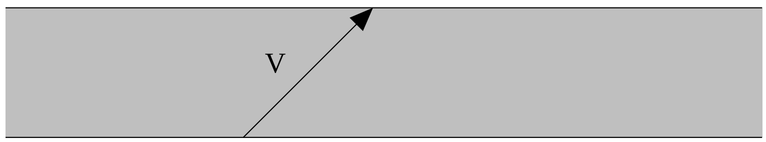

Figure 2.1: The vector V spans the strip.

The pair (Σ, V) defines a map $T : \boldsymbol{R}^2 \to \Sigma$, as follows.

$$T(p) = p + nV \tag{9}$$

Here $n \in \boldsymbol{Z}$ is the integer such that $p + nV \in \Sigma$. The map T is well-defined in the complement of a discrete infinite family of lines which are parallel to Σ. This family of lines contains the two lines of $\partial\Sigma$.

2.2 Outer Billiards and Pinwheel Maps

Here we recall the set-up in [**S4**]. Let P be a convex polygon with no parallel sides. Let ψ denote the second iterate of the outer billiards map on a convex polygon P.

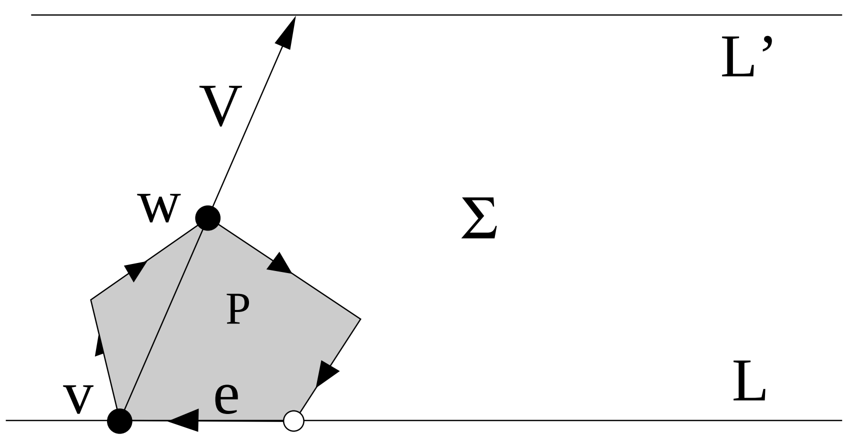

Figure 2.2: The strip associated to e.

We orient the edges of P so someone walking along an edge in the direction of the orientation would see P on the right. Given an edge e of P, we let L be the line extending e and we let L' be the line parallel to L so that the vertex w of P that lies farthest from L is equidistant from L and L'. We associate to e the pair (Σ, V), where Σ is the strip bounded by L and L', and $V = 2(w - v)$. See Figure 1.2.

We order the strips according to their slopes, so that one generally turns counter clockwise when changing from Σ_i to Σ_{i+1}. This ordering typically does not coincide with the cyclic ordering on the edges. The corresponding composition $T = T_n \circ ... \circ T_1$ of strip maps is a pinwheel system. To describe the connection between T and outer billiards, we first work outside some large compact subset $K \subset \boldsymbol{R}^2$. Suppose we start with a point $p_1 \in \Sigma_1$. Then $\psi^k(p_1) = p_1 + kV_2$ for $k = 1, 2, 3...$ This general rule continues until we reach an an exponent k_1 such that $p_2 = \psi^{k_1}(p_1) \in \Sigma_2$. Then we have $\psi^k(p_2) = p_2 + kV_3$ for $k = 1, 2, 3,$ until we reach an exponent k_2 such that $p_3 = \psi^{k_2}(p_2) \in \Sigma_3$. And so on. See Figure 1.3. We eventually reach a point $p_{n+1} \in \Sigma_1$, and the map $p_1 \to p_{n+1}$ is the first return map.

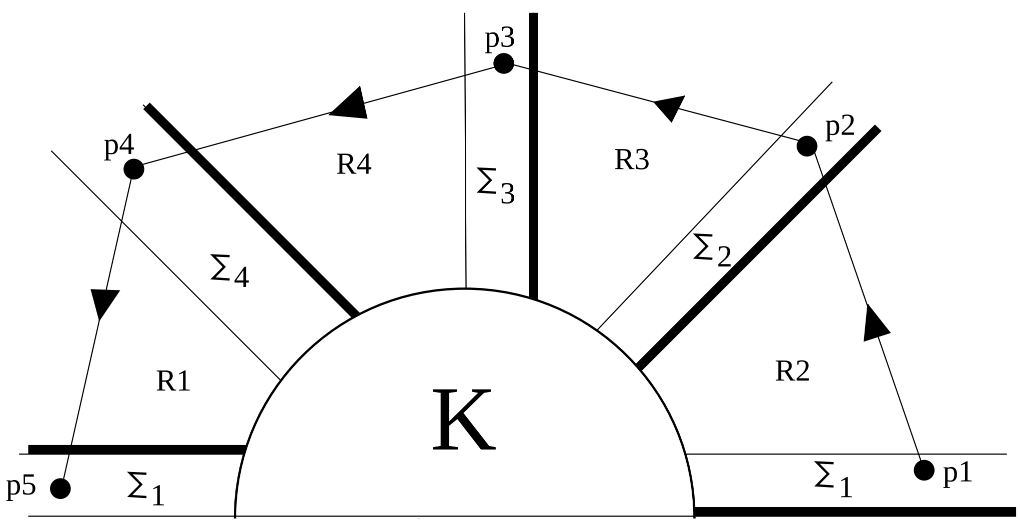

Figure 2.3: Far from the origin.

The connection between the first return map and ψ, for orbits *far away* from the polygon, appears in almost every paper on polygonal outer billiards. However, for points which start out near P, the connection is much less clear. Our main result in [**S4**] shows that the correspondence between the pinwheel map and outer billiards works well regardless of whether one starts near P or far away. We prove the following result in [**S4**].

Theorem 2.1 *There is a canonical bijection between the set of unbounded orbits of ψ and the set of unbounded orbits of T. The bijection is such that the ψ-orbit O corresponds to the T-orbit which agrees with $O \cap \Sigma_1$ outside a compact set. In particular, outer billiards on P has unbounded orbits if and only if T has unbounded orbits.*

Remarks: (i) We generally work with the map T^2, because T^2 turns out to be uniformly close to the identity map. T^2 encodes the effect of circulating all the way around P whereas T encodes the effect of going halfway around. The reader will see our preference for T^2 in the way we make our construction in the next section.
(ii) Let ψ' denote the outer billiards map, so that $\psi = (\psi')^2$. Each unbounded ψ'-orbit is the union of two unbounded ψ orbits. The corresponding unbounded T-orbits are related by a certain rotation of the strip Σ_1.

2.3 The Main Construction

Let $(\Sigma_1, V_1), ..., (\Sigma_{2n}, V_{2n})$ denote the strip data above, repeated out twice. Let

$$T_k : \Sigma_{k-1} \to \Sigma_k \tag{10}$$

be the corresponding strip map.

For the purpose of getting the signs right when we define certain maps, we assume that $\Sigma_1 = \mathbf{S}$ and $\Sigma_2, ..., \Sigma_n$ all have positive slope, as in Figure 2.3. However, after we define our maps, we will not insist on this normalization. For $k = 1, ..., n$, we define map

$$A_{k,\pm} : \Sigma_k \to \mathbf{S}, \tag{11}$$

by the following rules.

- $A_{k,\pm}$ is area preserving, orientation preserving, and maps points with large positive y-coordinate to points with large positive x-coordinate.

- $A_{k,\pm}$ maps the parallelogram $\Sigma_k \cap \Sigma_{k\pm1}$ to a rectangle, and the head point of V_k to the origin.

Let $\rho(x,y)=(-x,-y)$ be reflection about the origin. We define

$$A_{n+k,\pm}=\rho\circ A_{k,\pm}, \tag{12}$$

$$R_k=A_{k+1,-}\circ T_k\circ A_{k,+}^{-1}; \qquad S_k=A_{k+1,+}\circ(A_{k+1,-})^{-1} \tag{13}$$

Now that we have defined these maps, we drop the assumption about the slopes of the strips. In general, the maps are defined in such a way that the whole construction is natural under affine conjugation.

Lemma 2.2 *S_k is an affine shear, as in Equation 2.*

Proof: The maps $A_{k+1,\pm}$ are both area preserving, orientation preserving, sense preserving affine bijections from Σ_{k+1} to $\mathbf{S}$, and they both map the same point to the origin. From this description, it is clear that S_k has the equation given in Equation 2. The only thing that remains to prove that $b>0$. That is, S_k shears points in $\mathcal{S}$ with positive y-coordinate to the left.

To understand what is going on, we take $k=1$. Now we normalize so that Σ_1 is horizontal and Σ_2 is vertical, and the positive senses of these strips go along the positive coordinate axes. This situation forces Σ_3 to have negative slope. $\Sigma_1\cap\Sigma_2$ is the thickly drawn square and $\Sigma_2\cap\Sigma_3$ is the shaded parallelogram. AThe significant feature here is that the left side of the shaded parallelogram lies above the right side. This fact translates into the statement that $b>0$ in Equation 2. ♠

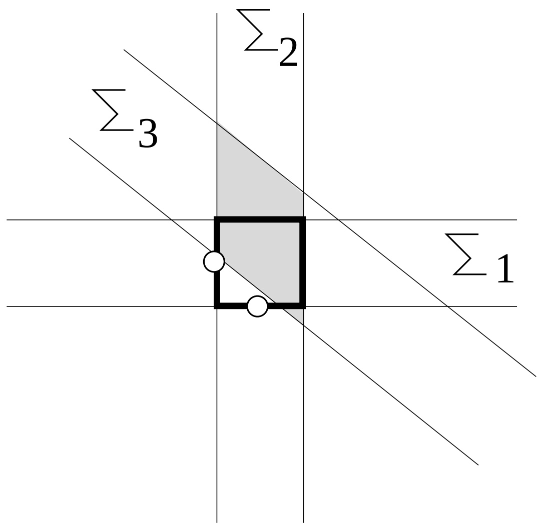

Figure 2.3: Placement of the strips

Lemma 2.3 *R_k is a quarter turn map.*

Proof: Our proof will also identifies the parameters of R_k. The parameter r_k is just the area of $\Sigma_k \cap \Sigma_{k-1}$. We will consider the case $k = 1$. We first discuss how our construction interacts with affine transformations. Let Δ be an affine transformation, which expands areas by δ. Let R_1' be the map associated to $\Delta(P)$. We have

$$R_1' = D_\delta \circ R_1 \circ D_\delta^{-1}, \qquad D_\delta(x, y) = (\delta x, y).$$

Thanks to this equation, it suffices to prove our result for any affine image of P. We normalize by an affine transformation so that

$$\Sigma_1 = \boldsymbol{R} \times [-1/2, 1/2], \qquad \Sigma_2 = [-1/2, 1/2] \times \boldsymbol{R}, \qquad V_2 = (-1, 1). \quad (14)$$

In this case, $A_{1,+}$ is the identity and $A_{2,-}$ is the clockwise order 4 rotation about the origin. Figure 2.4 shows the action of T_2 on Σ_1.

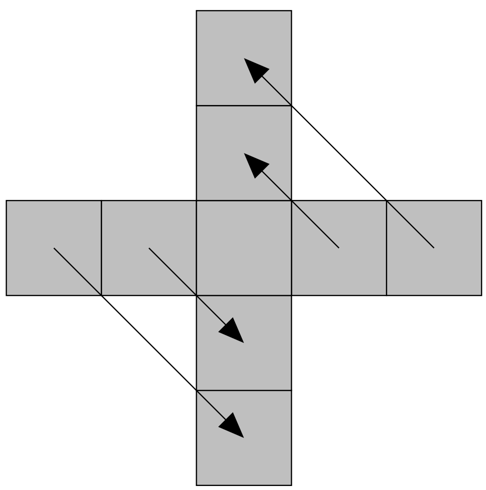

Figure 2.4: Action of T_2.

From Figure 2.4, and from the description of $A_{1,+}$ and $A_{2,-}$, we see that there is a tiling $\mathcal{S}$ of $\mathbf{S}$ by unit squares and R_1 gives a clockwise quarter turn to each unit square. To finish the proof, we just have to see that $\mathcal{S}$ is one of the two special tilings discussed in §1.3.

Let e_j be the edge of P that lies in $\partial\Sigma_j$. Let $|V_j|$ is the segment underlying the vector V_j One basic principle we use in our analysis is that $e_1, |V_1|, |V_2|$ make the edges of a triangle. Call this the *triangle property*.

Let c_j be the head of V_j. We have

$$R_{1,+}(c_1) = (0,0) = R_{2,-}(c_2). \tag{15}$$

There are two cases to consider. Suppose that c_1 is not incident to e_2. By the triangle property, c_1 is incident to V_2. Hence either $c_1 = c_2$ or c_1 is the tail vertex of V_2. But the tail vertex of V_2 is incident to e_2. This proves that $c_1 = c_2$. This situation implies that the common point $c = c_1 = c_2$ is the center of $\Sigma_1 \cap \Sigma_2$. From this information, and Equation 15, we conclude that $\mathcal{S}$ is Tiling 1. Suppose that c_1 is incident to e_2. Then c_1 lies on the centerline of Σ_1 and on the boundary of Σ_2. Hence c_1 is the midpoint of an edge of $\Sigma_1 \cap \Sigma_2$. Hence, the origin is the midpoint of an edge of a tile in $\mathcal{S}$. Hence $\mathcal{S}$ is Tiling 2. ♠

We define

$$\mathcal{T}_P = S_n \circ R_n \circ \cdots \circ S_1 \circ R_1. \tag{16}$$

By construction, $\mathcal{T}_P$ is a QTC.

Let T be the map from Theorem 1.1. By construction

$$T = \rho \circ \mathcal{T}_P; \qquad T^2 = S_{2n} \circ R_{2n} \circ \cdots \circ S_1 \circ R_1. \tag{17}$$

Lemma 2.4 *$\mathcal{T}_P^2$ is finitary.*

Proof: We have $\mathcal{T}_P^2 = T^2$. The map T^2 is evidently a piecewise translation. We just need to prove that the set $\{T^2(p) - p|\ p \in S\}$ is finite. There is a sequence of numbers $m_1, ..., m_{2n}$ such that

$$T^2(p) - p = \sum_{i=1}^{2n} m_i V_i = \sum_{i=1}^{n} (m_i - m_{i+n}) V_i$$

Here $V_1, ..., V_n$ are the vectors that arise in the strip maps, and we are setting $V_{i+n} = -V_i$. The number m_j refers to the analysis of Figure 2.3. Here m_j is the number of iterates of ψ needed to carry the iterate lying in Σ_{j-1} to the iterate lying in Σ_j.

Far from the origin, the portion of the ψ-orbit of p, going from p to $T^2(p)$, lies within a uniformly bounded distance of a centrally symmetric $2n$-gon. The point is that all the strips come within a uniform distance of the origin. From this property, we see that there is a uniform bound to $|m_i - m_{i+n}|$ for all i. Hence, there are only finitely many choices for $T^2(p) - p$. ♠

2.4 The End of the Proof

Now we finish the proof of Theorem 1.1. The end of the proof is really just tedious book-keeping. Using the fact that $\mathcal{T}_P$ commutes with ρ and that $T = \rho \circ \mathcal{T}_P$, we conclude that ρ commutes with T and hence $T^2 = (\rho \circ T)^2$.

Now we define a map $\Theta : U(P) \to U(\mathcal{T}_P)$. Let $O' \in U(P)$ be an unbounded outer billiards orbit. We can write $O' = O_1' \cup O_2'$, the union of even and odd iterates. Here both O_1' and O_2' are unbounded ψ-orbits. By Theorem 2.1, there are unbounded T-orbits O_1 and O_2 so that $O_k = O_k' \cap S$ outside a compact set. We write $O_1 = \{...p_1, p_3, p_5, ...\}$ and $O_2 = \{...p_2, p_4, p_6, ...\}$.

We define $\Psi(O')$ to be the equivalence class of.

$$..., p_2, \rho(p_4), p_6, \rho(p_8), ... \tag{18}$$

Lemma 2.5 Ψ *is well defined.*

Proof: Ψ is well defined because the orbit $..., p_1, \rho(p_3), p_5, \rho(p_7), ...$ ie equivalent to the orbit in Equation 18 because there is some index n such that $p_{n+1} = \rho(p_n)$. We just choose n so that p_n has very large negative x-coordinate. ♠

Lemma 2.6 Θ *is surjective.*

Proof: If $\{q_i\}$ is an unbounded orbit for $\mathcal{T}_P$, then $...q_1, \rho(q_2), q_3, \rho(q_4), ...$ is an unbounded orbit of T. Call this orbit O_1. There is some unbounded outer billiards orbit O' so that $O' = O_1' \cup O_2'$, as above, and $O_1' \cap S = O_1$ outside a compact set. Hence $\Theta(O)$ is the orbit $\{q_i\}$. ♠

Lemma 2.7 Θ *is injective.*

Proof: Suppose that $\overline{O}'$ is another unbounded orbit such that $\Theta(O') = \Theta(\overline{O}')$. Using the notation from above, we write $\overline{O}' = \overline{O}_1' \cup \overline{O}_2'$. The corresponding T-orbit $...\overline{p}_2, \rho(\overline{p}_4), \overline{p}_6, \rho(\overline{p}_8)...$ gets mapped to the same equivalence class as the orbit in Equation 18. But then either there is some $j \in \boldsymbol{Z}$ so that $p_{k+2j} = \overline{p}_k$ or $p_{k+2j} = \rho(\overline{p}_k)$ for all k. In the first case, obviously $O_2 = \overline{O}_2$. In the second case, $O_2 = \psi'(\overline{O}_2)$ for the same reason discussed in the previous lemma. So, either $O_2 = \overline{O}_2$ or else $O_2 = \overline{O}_1$. But then $O = O'$ by Theorem 2.1. ♠

3 The Compactification

3.1 Affine Pets Redefined

For the purposes of proving Theorem 1.2, we define affine PETs somewhat differently than we did in the introduction. We say that an almost everywhere defined map $f : \widehat{\mathbf{S}} \to \widehat{\mathbf{S}}$ is an *affine PET* if

1. f is defined except on a finite number of codimension 1 sub-tori.

2. f is injective and locally affine.

3. The linear part of f is independent of the point where it is computed.

4. f^{-1} also has Properties 1-3.

If f is an affine PET in this sense, we can subdivide S into convex polytopes, say $\widehat{\mathbf{S}} = \sqcup P_i$, so that f is defined on the interior of each P_i. Then we let $Q_i = f(P_i)$. Then $\widehat{\mathbf{S}} = \sqcup Q_i$, and f is an affine PET in the sense defined in the introduction. Conversely, an affine PET in the sense of the introduction satisfies the properties above. What is nicer about the above definition is that we don't need to explicitly divide the ambient torus into polytopes. For instance, an ordinary affine map of the torus is considered an affine pet. Also, the maps mentioned in Theorem 1.3 count as affine PETs.

We say that a dense open set $U \subset \widehat{\mathbf{S}}$ is an *invariant domain* for an affine PET f if f is entirely defined on U and $f(U) = U$.

It is convenient to let $\boldsymbol{T}^d = \boldsymbol{R}^d/\boldsymbol{Z}^d$ denote the unit torus made from the first d-coordinates of $\boldsymbol{R}^{n+1}$. Thus $\widehat{\mathbf{S}} = \boldsymbol{T}^{n+1}$.

3.2 The Map

Let $\mathcal{T}$, $\mathbf{S}$ and $\widehat{\mathbf{S}}$ be as in Theorem 1.2. We define $\Psi : \mathbf{S} \to \widehat{\mathbf{S}}$ by the formula

$$\Psi(x, y) = (\psi(x), [y]); \qquad \psi(x) = \left[\frac{x}{r_1}, ..., \frac{x}{r_n}\right]. \tag{19}$$

Here $[p]$ denotes the image of the point p in the torus.

Lemma 3.1 Ψ *is injective if and only if* $\mathcal{T}$ *is not quasi-rational.*

Proof: Ψ is not injective iff there are numbers x_1, x_2 such that $(x_1 - x_2)/r_i \in \boldsymbol{Z}$ for all i, which is true iff $\alpha_i = r_n/r_i \in \boldsymbol{Q}$ for all i. ♠

Let d be the dimension of the $\boldsymbol{Q}$-vector space $\boldsymbol{Q}(\alpha_1, \ldots, \alpha_{n-1})$.

Lemma 3.2 *If $d = n$ then $\Psi(\mathbf{S})$ is dense in $\widehat{\mathbf{S}}$.*

Proof: It suffices to prove that $\psi(\boldsymbol{R})$ is dense in $\boldsymbol{T}^n$. Let $\pi : \boldsymbol{R}^n \to \boldsymbol{T}^n$ be projection. Let X denote the closure of $\psi(R)$ in $\boldsymbol{T}^n$. Note that X is an abelian group. If $X \neq \boldsymbol{T}^n$, then X is a lower dimensional flat sub-torus of $\boldsymbol{T}^n$, and the connected component $\widetilde{X}$ of $\pi^{-1}(X)$ through the origin is a rational subspace. Hence, there is an integer linear map $f : \boldsymbol{R}^n \to \boldsymbol{R}$ which vanishes on $\widetilde{X}$, and f gives a rational relation amongst $\alpha_1, ..., \alpha_{n-1}, 1$. ♠

Lemma 3.3 $\dim(\mathbf{S}^*) = d + 1$*, where d is the dimension of $\boldsymbol{Q}(\alpha_1, ..., \alpha_{n-1})$.*

Proof: It suffices to prove that the closure X of $\psi(\boldsymbol{R})$ in $\boldsymbol{T}^n$ has dimension d. Permuting the coordinates, it suffices to consider the case when $\alpha_{n-d+1}, ..., \alpha_{n-1}, 1$ are independent over $\boldsymbol{Q}$ and α_j is a rational combination these last d variables for all $j \leq n - d$. Let $\pi : \boldsymbol{T}^n \to \boldsymbol{T}^d$ be projection onto the last d coordinates. By the previous result, $\pi(X) = \boldsymbol{T}^d$. To prove that $\dim(X) = d$ it suffices to prove that $X \cap \pi^{-1}(0, ..., 0)$ consists of finitely many points. Let p be a point in this intersection. We will show that the first coordinate of p can only take on finitely many values. The same argument works for the remaining coordinates.

We have some integer relation

$$c_1\alpha_1 = c_{n-d+1}\alpha_{n-d+1} + \ldots + c_{n-1}\alpha_{n-1} + c_n. \tag{20}$$

Multiplying through by r_n we have

$$\frac{c_1}{r_1} = \frac{c_{n+d-1}}{r_{n+d-1}} + \ldots \frac{c_n}{r_n}. \tag{21}$$

Suppose $x \in \boldsymbol{R}$ is such that $\pi \circ \psi(x)$ is close to $(0, ..., 0)$. Then x/r_j is close to an integer for $j = n - d + 1, ..., n$. But then $c_j x/r_j$ is also close to an integer for $j = n - d + 1, ..., n$. But then $c_1 x/r_1$ is close to an integer. This argument shows that the first coordinate of and point of $F \cap \pi^{-1}(0, ..., 0)$ has the form $[k/c_1]$ for some $k \in \{1, ..., c_1\}$. In particular, this is a finite set of possibilities. ♠

3.3 Extending The Component Maps

A QTC is the composition of two kinds of maps. In this section we treat each of these maps in isolation.

Lemma 3.4 *Let S be a shear of* $\mathbf{S}$ *in Equation 2. There is an affine PET* $\Psi \circ S = \widehat{S} \circ \Psi$*, where*

$$\widehat{S}([x_1, ..., x_n, y]) = \Big[x_1 - \frac{s}{r_1} y, ..., x_n - \frac{s}{r_n} y, y\Big]. \tag{22}$$

$\widehat{S}$ *is an affine PET, and* $\boldsymbol{T}^n \times (-1/2, 1/2) \subset \widehat{\mathbf{S}}$ *is an invariant domain for* $\widehat{S}$*. The linear part of* $\widehat{S}$ *is given by the matrix*

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & -s/r_1 \\ 0 & 1 & 0 & \cdots & -s/r_2 \\ 0 & 0 & 1 & \cdots & -s/r_3 \\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{23}$$

Proof: We have $S(x, y) = (x - sy, y)$. A direct calculation shows that $\Psi \circ S = \widehat{S} \circ \Psi$ for the map $\widehat{S}$ given above. Once we have the map $\widehat{S}$, the given domain is clearly an invariant domain. ♠

Lemma 3.5 *Let* $R_n = R_{q_n, r_n}$*. There exists an affine PET* $\widehat{R}_n : \widehat{\mathbf{S}} \to \widehat{\mathbf{S}}$ *such that* $\Psi \circ R_n = \widehat{R}_n \circ \Psi$*. The linear part of* $\widehat{R}_n$ *is given by the matrix*

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & -r_n/r_1 & r_n/r_1 \\ 0 & 1 & 0 & \cdots & -r_n/r_2 & r_n/r_2 \\ 0 & 0 & 1 & \cdots & -r_n/r_3 & r_n/r_3 \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix} \tag{24}$$

Define

$$X_{0,n} = \boldsymbol{T}^{n-1} \times (-1/2, 1/2) \times (-1/2, 1/2); \quad X_{1,n} = \boldsymbol{T}^{n-1} \times (0, 1) \times (-1/2, 1/2) \tag{25}$$

When $q_n = k$*, the set* $X_{k,n}$ *is an invariant domain for* $\widehat{R}_n$*.*

Proof: Suppose $q_n = 0$. Let

$$\Psi^*(x,y) = \left[\frac{x^*}{r_1}, \cdots, \frac{x^*}{r_{n-1}}, \frac{x}{r_n}, y\right], \qquad x^* = r_n \mathrm{int}\left(\frac{x}{r_n}\right).$$

Here $\mathrm{int}(x)$ is the integer nearest x. If $(x_2, y_2) = R_n(x_1, y_1)$, then $x_2^* = x_1^*$. Hence $\Psi^* \circ R_n = F \circ \Psi^*$, where F does nothing to the first $n-1$ coordinates and, with respect to the last two coordinates, acts as an order 4 clockwise rotation fixing $[0,0]$. That is,

$$F([x_1, ..., x_{n-1}, x_n, y]) = [x_1, ..., x_{n-1}, y + (q_n/2), -x_n + (q_n/2)]. \qquad (26)$$

On $X_{0,n}$, we have $\Psi = Y \circ \Psi^*$, where

$$Y([x_1, \cdots, x_{n-1}, x_n, y]) = \left[x_1 + \frac{r_n}{r_1} x_n, \cdots, x_{n-1} + \frac{r_n}{r_{n-1}} x_n, x_n, y\right]. \qquad (27)$$

The set $X_{0,n}$ evidently is an invariant domain for both Y and F. The map

$$\widehat{R}_n = Y \circ F \circ Y^{-1}, \qquad (28)$$

has all the desired properties. A short exercise in matrix multiplication shows that the linear part of $\widehat{R}_n$ has the form given in Equation 24.

When $q_n = 1$ we use the floor function in place of the nearest integer function when defining Ψ^*. ♠

The same result as above holds for R_{q_j, n_j}. The only difference is that the roles played by the indices j and n are swapped. For instance, the linear part of $\widehat{R}_1$ is given by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ -r_1/r_2 & 1 & 0 & \cdots & 0 & r_1/r_2 \\ -r_1/r_3 & 0 & 1 & \cdots & 0 & r_1/r_3 \\ \cdots & & & & & \\ -r_1/r_n & 0 & 0 & \cdots & 1 & r_1/r_n \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad (29)$$

and the invariant domain is obtained from one of the domains in Equation 25 by permuting the 1st and nth coordinates.

3.4 The Composition

We now define

$$\widehat{\mathcal{T}} = \widehat{S}_{s_n} \circ \widehat{R}_{q_n,r_n} \circ \cdots \circ \widehat{S}_{s_1} \circ \widehat{R}_{q_1,r_1}. \tag{30}$$

The composition of affine pets is an affine PET. so $\widehat{\mathcal{T}}$ is an affine PET. By construction, $\Psi \circ \mathcal{T} = \widehat{\mathcal{T}} \circ \Psi$.

Let $\mathbf{S}^*$ denote the closure of $\Psi(\mathbf{S})$ in $\widehat{\mathbf{S}}$. Now we suppose that $\mathcal{T}^k$ is finitary for some exponent k. We will prove that the restriction of $\widehat{\mathcal{T}}^k$ to $\mathbf{S}^*$ is an ordinary PET. For ease of notation, we assume that $k = 1$. The proof works the same way regardless of exponent.

We just need to show that $\widehat{\mathcal{T}}$ is a local translation. Suppose that $\widehat{p} \in \mathbf{S}^*$ and $\{\widehat{p}_n\}$ is a sequence of points in $\mathbf{S}^*$ converging to $\widehat{p}$. We want to show that

$$\widehat{\mathcal{T}}(\widehat{p}) - p = \widehat{\mathcal{T}}(\widehat{p}_n) - p_n, \tag{31}$$

for all n sufficiently large. Since $\Psi(\mathbf{S})$ is dense in $\mathbf{S}^*$ and the linear part of $\widehat{\mathcal{T}}$ is independent of point, it suffices to consider the case when $\widehat{p} = \Psi(p)$ and $\widehat{p}_n = \Psi(p_n)$ for some $p \in \mathbf{S}$ and some sequence $\{p_n\}$ in $\mathbf{S}$. Note that $\{p_n\}$ need not be a convergent sequence in $\mathbf{S}$.

Lemma 3.6 *Setting $V_s = \mathcal{T}(s) - s$ for any $s \in S$, we have*

$$\widehat{\mathcal{T}}(p) - p = \Psi(V_p), \qquad \widehat{\mathcal{T}}(p_n) - p_n = \Psi(V_{p_n}). \tag{32}$$

Proof: We have

$$\widehat{\mathcal{T}}(\widehat{p}) - \widehat{p} = \widehat{\mathcal{T}} \circ \Psi(p) - \Psi(p) = \Psi \circ \mathcal{T}(p) - \Psi(p) = \Psi(V_p) \tag{33}$$

The last equality comes from the fact that $\Psi(V+W) = \Psi(V)+\Psi(W)$ whenever V, W, and $V+W$ all belong to S. Here we are taking $V = V_p$ and $W = p$. The same argument works for p_n. ♠

We now observe the following properties.

1. By continuity, $\Psi(V_{p_n}) \to \Psi(V_p)$ as $n \to \infty$.
2. Since $\mathcal{T}$ is finitary, there is a uniform upper bound to $|V_{p_n}|$.
3. Ψ is injective.

It follows from these properties that $V_{p_n} \to V_p$. But $\mathcal{T}$ is finitary. Hence $V_{p_n} = V_p$ for n large. But then $\Psi(V_p) = \Psi(V_{p_n})$ for n large. This fact combines with Equation 32 to establish Equation 31 for n large.

3.5 Discussion

Here we discuss some features of the construction above. To check whether some iterate $\widehat{\mathcal{T}}^k$ is a PET, we just have to see that the suitable composition of the matrices in Lemmas 3.4 and 3.5 is the identity. Using this principle, we can establish some basic facts.

Lemma 3.7 *$\widehat{\mathcal{T}}$ is never a piecewise isometry on all of $\widehat{\mathbf{S}}$.*

Proof: Let $L(X)$ be the linear part of the map X. The composition

$$L(\widehat{S}_{n-1}) \circ L(\widehat{R}_{n-1}) \circ \cdots \circ L(\widehat{S}_1) \circ L(\widehat{R}_1)$$

acts as the identity on the vector $(0, ..., 0, 1, 0)$. Hence, the corresponding matrix has $(0, ..., 0, 1, 0)^t$ as its nth column. On the other hand, the last row of

$$L(\widehat{S}_n) \circ L(\widehat{R}_n)$$

is $(0, ..., -1, 0)$ as its last row. Hence, the linear part of $\widehat{\mathcal{T}}$ is represented by a matrix that has -1 in an off-diagonal position. ♠

This result combines with Theorem 1.2 to prove the following result.

Corollary 3.8 *If $\mathcal{T}$ is finitary, then $\Psi(\mathbf{S})$ is not dense in $\widehat{\mathbf{S}}$.*

The nicest case of Theorem 1.2 arises when Ψ does have a dense image. In this case, $\mathcal{T}$ is not finitary. However, often it happens that $\mathcal{T}^2$ is finitary. In this case, $\widehat{\mathcal{T}}^2$ is a PET that is defined on all of $\widehat{\mathbf{S}}$.

4 The Structure of the Compactification

4.1 The Singular Directions

In this chapter we prove Theorem 1.3. Let $\mathcal{H}$ denote a finite union of n-dimensional linear subspaces of $\boldsymbol{R}^{n+1}$ We say that $\mathcal{H}$ is a *complete set* for the map $\widehat{\mathcal{T}}$ if $\widehat{\mathcal{T}}$ is defined on $\widehat{\mathbf{S}} - X$, where X is a finite union of codimension 1 flat tori and each element of X is parallel to some element of $\mathcal{H}$.

In this section we will produce a complete set $\mathcal{H}$ with $n+1$ members. This is a first step towards proving Theorem 1.3 because the map $[X_1, X_2, I]$ discussed in Theorem 1.3 have complete sets with $n+1$ members.

Let Π_k denote the hyperplane given by the equation $x_k = 0$. To keep our notation consistent with the previous chapter, we say that Π_{n+1} is the hyperplane given by $y = 0$. Let $L(F)$ denote the linear part of the affine map F.

Lemma 4.1 *A complete set for $\mathcal{T}$ is given by*

1. $\{\Pi_1, \Pi_{n+1}\}$.
2. $L(\widehat{R}_1)^{-1}(\Pi_{n+1})$.
3. $L(\widehat{R}_1)^{-1}L(\widehat{S}_1)^{-1}(\{\Pi_2, \Pi_{n+1}\})$,
4. $L(\widehat{R}_1)^{-1}L(\widehat{S}_1)^{-1}L(\widehat{R}_2)^{-1}(\Pi_{n+1})$,
5. $L(\widehat{R}_1)^{-1}L(\widehat{S}_1)^{-1}L(\widehat{R}_2)^{-1}L(\widehat{S}_2)^{-1}(\{\Pi_3, \Pi_{n+1}\})$,

and so on.

Proof: In view of Lemma 3.4, the hyperplane Π_{n+1} is a complete set for $\widehat{S}_j$. In view of Lemma 3.5 (and the remarks after it), the two hyperplanes $\{\Pi_k, \Pi_{n+1}\}$ form a complete set for $\widehat{R}_k$. If the map $\widehat{\mathcal{T}}$ is not defined on some point p, then one of the compositions

$$F_k = \widehat{R}_k \circ \widehat{S}_{k-1} \circ \cdots \circ \widehat{S}_1 \circ \widehat{R}_1, \qquad G_k = \widehat{S}_k \circ \widehat{R}_k \circ \cdots \circ \widehat{S}_1 \circ \widehat{R}_1 \qquad (34)$$

is undefined at p but all shorter compositions are defined. But then either $F_k(p)$ lies in the boundary of the invariant domain for $\widehat{R}_k$ or $G_k(p)$ lies in the boundary of the invariant domain for $\widehat{S}_k$. But then p lies in a hypersurface parallel to one of the hyperplanes on our list. ♠

Lemma 4.2 *A complete list for $\mathcal{T}$ is given by $H_1, ..., H_{n+1}$, where $H_1 = \Pi_1$ and $H_{n+1} = \Pi_{n+1}$ and*

$$H_{k+1} = L(\widehat{R}_1)^{-1} \circ ... \circ L(\widehat{S}_k)^{-1}(\Pi_{k+1}), \qquad k = 1, ..., n-1. \tag{35}$$

Proof: Note that there are about $3n$ hyperplanes listed in Lemma 4.1 whereas we are claiming that $n+1$ hyperplanes suffices. The idea here is just to eliminate redundancies. First, we have

$$L(\widehat{R}_k)^{-1}(\Pi_{n+1}) = \Pi_k. \tag{36}$$

Therefore, each hyperplane listed on line $2k$ of Lemma 4.1 is contained in one of the hyperplanes listed on line $2k-1$.

Second, we have

$$L(S_k)^{-1}(\Pi_{n+1}) = \Pi_{n+1}, \qquad L(\widehat{R}_k)^{-1}(\Pi_k) = \Pi_{n+1}. \tag{37}$$

Therefore, the second hyperplane listed on line $2k+1$ of Lemma 4.1 is contained first hyperplane listed on line $2k-1$. For instance, taking $k = 2$, we have

$$L(\widehat{R}_1)^{-1}L(\widehat{S}_1)^{-1}L(\widehat{R}_2)^{-1}L(\widehat{S}_2)^{-1}(\Pi_{n+1}) =$$
$$L(\widehat{R}_1)^{-1}L(\widehat{S}_1)^{-1}L(\widehat{R}_2)^{-1}(\Pi_{n+1}) = L(\widehat{R}_1)^{-1}L(\widehat{S}_1)^{-1}(\Pi_2).$$

Upon eliminating all the redundancies, we get the advertised list. ♠

Let e_k denote the kth standard basis vector in $\boldsymbol{R}^{n+1}$. Let $H_k^\perp$ denote the normal to H_k.

Lemma 4.3 *The matrix whose rows are $H_1^\perp, ..., H_{n+1}^\perp$ has determinant 1.*

Proof: Let $M_k = L(\widehat{R}_1) \circ ... \circ L(\widehat{S}_{k-1})$. We have $H_n^\perp = (0, ..., 0, 1)$ and

$$H_k^\perp = (M_k^{-1})^t(e_k), \qquad k = 1, ..., n. \tag{38}$$

The maps $L(\widehat{R}_j)$ and $L(\widehat{R}_k)$ act trivially on $e_{j+1}, ..., e_n$. Hence M_k acts trivially on $e_{k+1}, ..., e_n$. Hence, rows $k, ..., n$ of the inverse transpose matrix $(M_k^{-1})^t$ coincide with the rows of the identity matrix. Hence

$$H_k^\perp = (*, \cdots, *, 1, 0, \cdots, 0, *), \qquad k = 1, ..., n. \tag{39}$$

The 1 appears in the kth slot and $(*)$ indicates an entry that we don't explicitly know. The lemma is immediate from this structure. ♠

4.2 The First Parallelotope

Let $X_1 \subset \boldsymbol{R}^{n+1}$ be the parallelotope consisting of vectors V such that

$$H_i^\perp \cdot V \in [-1/2, 1/2] \tag{40}$$

for all i.

Lemma 4.4 *X_1 is a fundamental domain for $\boldsymbol{Z}^{n+1}$.*

Proof: In view of Lemma 4.3, the set X_1 is a unit volume parallelotope. Let M be the matrix with rows $H_1^\perp, ..., H_{n+1}^\perp$. From the proof of Lemma 4.3 we have

$$M = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & * \\ * & 1 & 0 & \cdots & 0 & * \\ * & * & 1 & \cdots & 0 & * \\ \cdots \\ * & * & * & \cdots & 0 & 1 \end{bmatrix} \tag{41}$$

X_1 consists of those vectors $V \in \boldsymbol{R}^{n+1}$ such that $MV \in [-1/2, 1/2]^{n+1}$.

Since X_1 has unit volume, it suffices to show that the interior of X_1 does not intersect some integer translate of X_1. This happens if and only if there is some integer vector $V \in \boldsymbol{Z}^{n+1}$ such that $MV \in (0,1)^{n+1}$. This is clearly impossible given the form of M. ♠

Remark: Given the form of the matrix in Equation 41, we can also say that X_1 is the polytope bounded by the hyperplanes $H_k \pm 1/2e_k$.

Let $q = (q_1, ..., q_n)$ and $r = (\cdots)$ and $s = (\cdots)$ be the invariants for $\mathcal{T}$. Let $\pi : \boldsymbol{R}^{n+1} \to \widehat{\mathbf{S}}$ be projection. Let X_1^o be the interior of X_1. Let I be the affine map which fixes the vector $q/2$ and whose linear part coincides with the linear part of $\widehat{\mathcal{T}}$.

Lemma 4.5 *the map $\widehat{\mathcal{T}}$ is entirely defined on $\pi(X_1^o + q/2)$ and $\widehat{\mathcal{T}} = \pi \circ I \circ \pi^{-1}$ on $\pi(X_1^o + q/2)$ provided that π^{-1} is taken to have its range in $X_1^o + q/2$.*

Proof: We will give the proof in case $q = (0, ..., 0)$. In this case, I is simply the linear part of $\widehat{\mathcal{T}}$. The general case has essentially the same proof, and differs only in that we apply suitable translations to the basic objects.

Let A_k denote the open slab bounded by the hyperplanes $x_k = \pm 1/2$. Let B_k denote the open slab bounded by the parallel hyperplanes $H_k \pm \frac{1}{2}e_k$. By construction $X_1 = \bigcap H_k$. Also by construction,

$$L(\widehat{R}_1)^{-1} \circ \cdots \circ L(\widehat{S}_{k-1})^{-1}(A_k) = B_k. \tag{42}$$

Let ρ_k be the restriction of $L(\widehat{R}_k)$ to $A_k \cap A_{n+1}$. Likewise, let σ_k be the restriction of $L(\widehat{S}_k)$ to A_{n+1}. Given the description of the invariant domains for $\widehat{R}_k$ and $\widehat{S}_k$ in §3.3, we have

$$\widehat{R}_k = \pi \circ \rho_k \circ \pi^{-1}, \qquad \widehat{S}_k = \pi \circ \sigma_k \circ \pi^{-1}. \tag{43}$$

The right hand side is independent of the lift, as long as the range of π^{-1} is taken to be $A_k \cap A_{n+1}$ or A_{n+1} respectively.

Choose any point $p \in \pi(X_1^o)$. Let q_1 be the unique point in X_1^o such that $\pi(q_1) = p$. By construction $q_1 \in B_1 \cap ... \cap B_{n+1}$. But $A_1 = B_1$ and $A_{n+1} = B_{n+1}$ Hence $q_1 \in A_1 \cap A_{n+1}$. Since $q_1 \in A_1 \cap A_{n+1}$, the map ρ_1 is defined on q_1. Since ρ_1 preserves $A_1 \cap A_{n+1}$, we have $\rho_1(q_1) \in A_1 \cap A_{n+1}$. In particular $\rho_1(q_1) \in A_{n+1}$, and so σ_1 is defined on $\rho_1(q_1)$. Equation 43 now gives us

$$q_2 = \sigma_1 \circ \rho_1(q_1) \in A_2 \cap A_{n+1}, \qquad \pi(q_2) = \widehat{S}_1 \circ \widehat{R}_1(p). \tag{44}$$

Repeating the same argument with q_2 in place of q_1, we see that ρ_2 is defined on q_2 and σ_2 is defined on $\rho_2(q_2)$ and

$$q_3 = \sigma_2 \circ \rho_2(q_2) \in A_3 \cap A_{n+1}, \qquad \pi(q_3) = \widehat{S}_2 \cap \widehat{R}_2 \circ \widehat{S}_1 \circ \widehat{R}_1(p). \tag{45}$$

Continuing in this way, we produce points $q_4, ..., q_n$ such that

- $q_k \in A_k \cap A_{n+1}$.
- $\sigma_k \circ \rho_k$ is defined on q_k.
- $q_{k+1} = \sigma_k \circ \sigma_k(q_k)$.
- $\pi \circ q_{k+1} = \widehat{S}_k \circ \widehat{R}_k(\pi(q_k))$.

In particular, $\widehat{\mathcal{T}}$ is defined on p and

$$\widehat{\mathcal{T}}(p) = \pi(q_n) = \pi \circ \sigma_n \circ \cdots \circ \rho_1 \circ \pi^{-1}(q_1) = I \circ \pi^{-1}(p). \tag{46}$$

Hence $\widehat{\mathcal{T}}$ is completely defined on $\pi(X_1^o)$ and $\widehat{\mathcal{T}} = \pi \circ I \circ \pi^{-1}$ on $\pi(X_1^o)$. ♠

4.3 The Second Parallelotope

Let $X_2 = I(X_1)$.

Lemma 4.6 *X_2 is a fundamental domain for $\boldsymbol{Z}^{n+1}$.*

Proof: Again, we consider the case when $q = (0, ..., 0)$ for ease of exposition. The linear parts of $\widehat{R}_k$ and $\widehat{S}_k$ are orientation preserving and volume preserving maps. We also know that X_1 is a unit volume parallelotope and a fundamental domain for $\boldsymbol{Z}^{n+1}$. Since I is volume preserving, X_2 is also a unit volume parallelotope.

The map $\widehat{\mathcal{T}}$ is invertible. In particular, the restriction of $\widehat{\mathcal{T}}$ to X_1^o is injective. But this map equals $\pi \circ I \circ \pi^{-1}$. Hence $\pi : X_2 \to \widehat{\mathbf{S}}$ is also injective. This fact, together with the fact that X_2 has unit volume, shows that X_2 is in fact a fundamental domain. ♠

When $q = (0, ..., 0)$, Lemma 4.5 tells us that $\widehat{\mathcal{T}} = [X_1, X_2, I]$. In general, let $X_j' = X_j + q/2$ and let I' be the affine map which fixes $q/2$ and whose linear part is I. Lemma 4.5 tells us that $\widehat{\mathcal{T}} = \pi \circ I' \circ \pi^{-1}$ on the interior of $\pi(X_1')$. But $[X_1', X_2', I']$ is conjugate to $[X_1, X_2, I]$.

More precisely, let $\tau : \widehat{\mathbf{S}} \to \widehat{\mathbf{S}}$ be translation by $q/2$. Then

$$\tau \circ [X_1, X_2, I] \circ \tau^{-1} = [X_1', X_2', I'].$$

It is convenient to define the new map

$$\Psi_q = \tau^{-1} \circ \Psi = \Psi - q/2. \tag{47}$$

A short calculation tells us that

$$\Psi_q \circ \mathcal{T} = [X_1, X_2, I] \circ \Psi_q. \tag{48}$$

From this alternate point of view, the compactified system $[X_1, X_2, I]$ is independent of the q parameters. What changes with the q parameters is the map Ψ_q.

It is worth remarking that the map $p \to -p$ gives an involution on $\widehat{\mathbf{S}}$ having 2^{n+1} fixed points. Of these fixed points, 2^n are distinguished by the property that the last coordinate is $[0]$. The map Ψ_q maps the origin to one of these distinguished fixed points.

4.4 The Fixed Point Set

It only remains to prove that I pointwise fixes a codimension 2 subspace. It suffices to consider the case $q = (0, ..., 0)$. We call a QTC *good* if Ψ is dense in $\widehat{\mathbf{S}}$ and the linear part of $\mathcal{T}$ has 2 unequal eigenvalues.

Lemma 4.7 *The set of good QTCs is dense in* $\boldsymbol{R}^{2n}$.

Proof: The set of QTCs satisfying the second condition has full measure. All we need here is that there are no rational relations amongst the numbers $r_n/r_1, ..., r_n/r_{n-1}, 1$. The set of QTCs satisfying the first condition has the form $F^{-1}(0)$ where F is a polynomial function. The point is that the trace of the linear part of $\mathcal{T}$ is a polynomial in the variables, and we just need to avoid the trace value 2. If we can show that the set of QTCs satisfying the first condition is nonempty, then this set is open dense. Finally, the intersection of an open dense set with a full measure set is dense.

In case n is not divisible by 4, the QTC with parameters $r = (1, ..., 1)$ and $s = (0, ..., 0)$ has the desired properties. When n is divisible by 4, the QTC with parameters $r = (1, ..., 1, 2)$ and $s = (0, ..., 0)$ has the desired properties. ♠

The linear map I depends continuously on the QTC parameters, so it suffices to consider the case when $\mathcal{T}$ is good.

Lemma 4.8 *Let A be an area preserving affine map of $\boldsymbol{R}^2$ with unequal eigenvalues. Let $\epsilon > 0$ be given. Then there is some $\delta > 0$ such that the bound $\|A(p) - p\| < \delta$ implies that A has a fixed point within ϵ of p.*

Proof: This is a standard result. It suffices to consider the case when A is linear. But, as is well known, the only points which such a map almost fixes are near the origin. ♠

Recall that we have the triple (X_1, X_2, I) associated to $\mathcal{T}$. The map I certainly fixes the origin. There is some small ball $X_1' \subset X_1$ centered at the origin such that $I(X_1') \subset X_1$.

Say that a *net* of $\mathbf{S}$ is a subset of points such that every point of $\mathbf{S}$ is within some N of a point in the subset. Let $\Theta \subset \mathbf{S}$ denote the set of points $(x, y) \in \mathbf{S}$ such that $\Psi(x, y) \in X_1'$ and $\mathcal{T}$ fixes (x, y).

Lemma 4.9 Θ *is a net in* $\mathbf{S}$.

Proof: Associated to the quarter turn maps $R_1, ..., R_n$ there are rectangle tilings $\mathcal{R}_1, ..., \mathcal{R}_n$. For any $\epsilon > 0$ we can find a net $X \subset \mathbf{S}$ with the following property. Each $(x, y) \in X$ is within ϵ of a center of a rectangle from each tiling. If ϵ is small enough then $\mathcal{T}$ is defined on a ball of radius $\epsilon_0 > 0$ about (x, y) and moves (x, y) no more than ϵ. Here ϵ_0 is a universal constant that does not tend to 0 with ϵ. For such points, the distance from $\Psi(x, y)$ to $(0, ..., 0)$ tends to 0.

If we choose ϵ small enough then there is some $\epsilon_1 > 0$ such that $\mathcal{T}$ fixes a point (x', y') within ϵ_1 of each $(x, y) \in X$. This follows from Lemma 4.8 applied to the restriction of $\mathcal{T}$ to the ϵ_0-ball about (x, y). Here ϵ_1 tends to 0 with ϵ. If ϵ_1 is sufficiently small then $\Psi(x', y') \in X_1'$. The set of all such (x, y') forms the desired net. ♠

Let $\Pi' \subset X_1'$ denote the intersection of the $+1$ eigenspace of I with X_1'. By construction I fixes a point in X_1' if and only if that point lies in Π'. Therefore, $\Psi(\Theta) \subset \Pi'$. We will suppose that $\dim(\Pi) < n - 1$ and derive a contradiction.

Let $\Pi \subset \boldsymbol{R}^{n+1}$ denote the linear subspace spanned by tangent vectors to Π' the two vectors $d\Psi(1, 0)$ and $d\Psi(0, 1)$. Then Π is a proper linear subspace of $\boldsymbol{R}^{n+1}$ and has measure 0. By the ergodic theorem, the set

$$X = \Psi^{-1}(\Pi \cap \widehat{\mathbf{S}}) \tag{49}$$

has density 0 in $\mathbf{S}$. On the other hand, since Ψ is locally affine, X contains the ϵ neighborhood of Θ for some $\epsilon > 0$. Since Θ is a net, and $\epsilon > 0$, we see that X has positive density in $\mathbf{S}$. This is a contradiction. We conclude that $\dim(\Pi') \geq n - 2$. This is what we wanted to prove.

Remark: In case $\mathcal{T}$ comes from Theorem 1.1, the linear part of $\mathcal{T}$ is the reflection through the origin. In this case, I acts on the 2-dimensional set $d\Psi(\boldsymbol{R}^2)$ as reflection through the origin. This shows that I has two -1 eigenvalues. Since we already know that I has $n-1$ eigenvalues of value 1, we see that the complete list of eigenvalues of I must be $-1, -1, 1, ..., 1$.

4.5 Double Lattice PETs

We call $[X_1, X_2, I]$ *standard* if I has eigenvalues $-1,-1,1,...,1$. Consider the map

$$\Psi' = \Pi_1^{-1} \circ \Psi : \mathbf{S} \to X_1. \tag{50}$$

The map Ψ' is a piecewise affine map. After a bit of algebra, we get

$$\Psi' \circ \mathcal{T} = (F \circ I) \circ \Psi', \qquad F = \Pi_1^{-1} \circ \Pi_2. \tag{51}$$

The map F has a very simple description. Given a generic $p \in X_2$ we have $F(p) = p + v$, where $v \in \boldsymbol{Z}^{n+1}$ is the unique vector such that $p + v \in X_1$.

Let $\Lambda_1 = \boldsymbol{Z}^{n+1}$ and $\Lambda_2 = I(\boldsymbol{Z}^{n+1})$. Note that X_i is a fundamental domain for Λ_j for all pairs (i,j) and the involution I has the action $\Lambda_1 \leftrightarrow \Lambda_2$ and $X_1 \leftrightarrow X_2$.

Let $\phi : X_1 \cup X_2 \to X_1 \cup X_2$ be the map with the following definition.

- If $p \in X_1 \cap X_2$, then $\phi(p) = p$.
- If $p \in X_1 - X_2$ then $\phi(p) = p + \lambda_2 \in X_2$ for the unique choice of $\lambda_2 \in \Lambda_2$ which works.
- If $p \in X_2 - X_1$ then $\phi(p) = p + \lambda_1 \in X_1$ for the unique choice of $\lambda_1 \in \Lambda_1$ which works.

The partition for ϕ is

$$(\Lambda_1 X_1 \# \Lambda_2 X_2)|_{X_1 \cup X_2}. \tag{52}$$

Here $\Lambda_j X_j$ is the partition of $\boldsymbol{R}^{n+1}$ by Λ_j translates of X_j. The symbol $(\#)$ denotes the common refinement of the partitions. The inverse map ϕ^{-1} has the same construction, except with the roles of Λ_1 and Λ_2 reversed. So, the partition for ϕ^{-1} is

$$(\Lambda_2 X_1 \# \Lambda_1 X_2)|_{X_1 \cup X_2}. \tag{53}$$

ϕ commutes with I and

$$\Psi' \circ \mathcal{T} = (I \circ \phi)|_{X_1} \circ \Psi', \qquad \Psi' \circ \mathcal{T}^2 = \phi^2|_{X_1} \circ \Psi'. \tag{54}$$

We call ϕ a *double lattice PET*. If we are willing to use the piecewise affine map Ψ', we see that the second iterate of a standard QTC is compactified by a double lattice PET.

There is somewhat more structure. Let L_I denote the linear part of the map I. Recall that $H_1, ..., H_{n+1}$ are the hyperplanes parallel to the sides of X_1.

Lemma 4.10 $L_I(H_n) = H_{n+1}$ *and* $L_I(H_{n+1}) = H_n$.

Proof: We will prove the first statement. The second statement follows from the fact that L_I is an involution. Equation 35 tells us that

$$H_n = L(\widehat{R}_1)^{-1} \circ ... \circ L(\widehat{S}_{n-1})^{-1}(\Pi_n). \tag{55}$$

On the other hand

$$L_I = L(S_n) \circ ... \circ L(R_1). \tag{56}$$

Therefore

$$L_I(H_n) = L(S_n) \circ L(R_n)(\Pi_n) =^* L(S_n)\Pi_{n+1} = \Pi_{n+1} = H_{n+1}. \tag{57}$$

The starred equation comes from the explicit form of the matrix in Lemma 3.5. ♠

Corollary 4.11 X_1 *and* X_2 *have two pairs of parallel faces in common, one of which is the pair* $e_{n+1} = \pm 1/2$.

Proof: We know that X_1 is bounded by the hyperplanes $e_{n+1} = \pm 1/2$. It follows from Lemma 4.10 that both X_1 and X_2 are bounded by the hyperplanes

$$\{v|\ H_n^{\perp} \cdot v = \pm 1/2\}, \qquad \{v|\ H_{n+1}^{\perp} \cdot v = \pm 1/2\}, \tag{58}$$

This is to say that X_1 and X_2 have two pairs of parallel faces in common. ♠

In our description of double lattice pets, it seems that the first lattice Λ_1 is favored, because $\boldsymbol{Z}^{n+1}$ is a familiar lattice. However, we can always choose some linear transformation T and define

$$I' = T \circ I \circ T^{-1}, \qquad X'_k = T(X_k), \qquad \Lambda'_k = T(\Lambda_k). \tag{59}$$

We can choose T so that

$$I' = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -I_2 \end{bmatrix} \tag{60}$$

The corresponding lattices Λ'_1 and Λ'_2 are (in terms of familiarity) on an equal footing, as are the fundamental domains X'_1 and X'_2. Furthermore, the map I' is as simple as possible within its conjugacy class. The double lattice PET based on $[X'_1, X'_2, \Lambda'_1, \Lambda'_2]$ is linearly conjugate to $[X_1, X_2, \Lambda_1, \Lambda_2]$, but is more symmetric. Though we don't know much about X'_1 and X'_2 we can at least say that they have 2 pairs of parallel faces in common, and that I' swaps these 2 pairs.

5 The Arithmetic Graph

5.1 The Translation Vector

In this chapter we introduce a construction we call the *arithmetic graph*. We used this construction extensively when analyzing outer billiards on kites. Let (X_1, X_2, I) be as in the previous chapter. Recall that $\Pi_1^{-1} : \widehat{\mathbf{S}} \to X_1$ is the map which lifts points to $\boldsymbol{R}^{n+1}$ then translates by suitable integer vectors. Sometimes we will use the notation $\Lambda_1 = \boldsymbol{Z}^{n+1}$ and $\Lambda_2 = I(\Lambda_1)$.

Lemma 5.1 *Suppose that $[X_1, X_2, I]^2$ is defined on $p \in \widehat{\mathbf{S}}$. There is a unique vector $v_p \in \boldsymbol{Z}^{n+1}$ so that $\Pi_1^{-1}(p) + I(v_p) \in X_2$.*

Proof: Recall that

$$[X_1, X_2, I] = \Pi_2 \circ I \circ \Pi_1^{-1}.$$

Since $[X_1, X_2, I]$ is defined on p, the map Π_1^{-1} is defined on p. Since X_2 is a fundamental domain for Λ_2, there is some $w_p \in \Lambda_2$ so that so that $\Pi^{-1}(p) + w_p \in X_2$. We set $v_p = I(w_p)$. This shows the existence of v_p.

Suppose that v_p is not unique, and v_p' is another vector such that

$$\Pi_1^{-1}(p) + I(v_p') \in X_2. \tag{61}$$

Let

$$x = I(\Pi_1^{-1}(p)). \tag{62}$$

Applying I to Equation 61 (and also the equation for v_p) we have

$$x + v_p \in X_1, \qquad x + v_p' \in X_1. \tag{63}$$

Since X_1 is a fundamental domain for Λ_1, this situation is only possible if some integer translate of x lies in ∂X_1. Let $\Pi : \boldsymbol{R}^{n+1} \to \widehat{\mathbf{S}}$ denote projection. Note that $\Pi = \Pi_j$ on X_j. Since some integer translate of x lies in ∂X_1, we have

$$\Pi_2(x) = \Pi(x) \in \Pi(\partial X_1) = \Pi_1(\partial X_1). \tag{64}$$

But

$$\Pi_2(x) = [X_1, X_2, I](p). \tag{65}$$

In short, $[X_1, X_2, I](p) \in \Pi_1(\partial X_1)$. But then $[X_1, X_2, I]$ is not defined on $[X_1, X_2, I](p)$. This is a contradiction. ♠

Lemma 5.2 *Suppose that $[X_1, X_2, I]^2$ is defined on $p \in \widehat{\mathbf{S}}$. Then*

$$[X_1, X_2, I]^2(p) = p + I(v_p). \tag{66}$$

The addition takes place in $\widehat{\mathbf{S}}$.

Proof: We compute

$$[X_1, X_2, I]^2 = \Pi_2 \circ I \circ \Pi_1^{-1} \circ \Pi_2 \circ I \circ \Pi^{-1}. \tag{67}$$

Letting ϕ be the double lattice PET associated to (X_1, X_2, L), we have

$$\Pi_1^{-1} \circ \Pi_2 = \phi|_{X_2}, \qquad I \circ \phi|_{X_2} \circ I = \phi|_{X_1}. \tag{68}$$

Therefore

$$[X_1, X_2, L]^2 = \Pi_2 \circ \phi|_{X_1} \circ \Pi_1^{-1}. \tag{69}$$

Let $\Pi : \boldsymbol{R}^{n+1} \to \widehat{\mathbf{S}}$ denote projection. Note that $\Pi = \Pi_j$ on X_j. Hence

$$[X_1, X_2, L]^2(p) = \Pi_2 \circ \phi|_{X_1} \circ \Pi_1^{-1}(p) = \Pi_2(\Pi_1^{-1}(p) + I(v_p)) =$$

$$\Pi(\Pi_1^{-1}(p) + I(v_p)) =^* \Pi(\Pi_1^{-1}(p)) + \Pi(I(v_p)) = p + \Pi(I(v_p)). \tag{70}$$

The starred equality comes from the fact that Π is a homomorphism. If If we consider addition in $\widehat{\mathbf{S}}$, the two expressions

$$p + \Pi(I(v_p)), \qquad p + I(v_p)$$

coincide. ♠

The following result is not so important, but seems worth pointing out.

Lemma 5.3 *Suppose that $[X_1, X_2, I]^2$ is defined on $p \in \widehat{\mathbf{S}}$. Then the last coordinate of v_p is zero.*

Proof: Let $x = I(\Pi_1^{-1}(p))$ we have

$$x \in X_2, \qquad x + v_p \in X_1 - \partial X_1. \tag{71}$$

By Corollary 4.11, both X_1 and X_2 are bounded by the hyperplanes $e_{n+1} = \pm 1/2$. So, the last coordinate of x lies in $[-1/2, 1/2]$ and the last coordinate of $x + v_p$ lies in $(-1/2, 1/2)$. This forces the last coordinate of v_p to be 0. ♠

5.2 Definition of the Arithmetic Graph

Given an orbit $O = \{p_k\}$ we define Γ_O to be the lattice path in $\boldsymbol{Z}^{n+1}$ such that

$$\Gamma_O(m+1) - \Gamma_O(m) = v_{p_m}. \tag{72}$$

Here v_{p_m} is as in the previous section. We call Γ_O the *arithmetic graph* of the orbit. Γ_O is only defined up to integer translation.

There is another way to think about the arithmetic graph, in which we consider many orbits of the same time. Our construction depends on the choice of an *offset vector* $V_0 \in \boldsymbol{R}^{n+1}$. Define the set

$$S(V_0) = \{V_0 + I(\boldsymbol{Z}^{n+1})\} \mod \boldsymbol{Z}^{n+1}. \tag{73}$$

In view of Lemma 5.2, the countable set $S(V_0) \subset \widehat{\mathbf{S}}$ is invariant under the action of $[X_1, X_2, I]^2$. That is, $S(V_0)$ is partitioned into orbits of $[X_1, X_2, I]^2$.

Define $\mu : \boldsymbol{Z}^{n+1} \to S(V_0)$ by the equation

$$\mu(V) = I(V) \mod \boldsymbol{Z}^{n+1}. \tag{74}$$

Given $V, V' \in \boldsymbol{Z}^{n+1}$, we join V and V' by a directed edge if and only if

$$\mu(V') = [X_1, X_2, I]^2(\mu(V)). \tag{75}$$

This construction produces a directed graph $\Gamma(V_0)$ whose vertex set is $\boldsymbol{Z}^{n+1}$. Each component of $\Gamma(V_0)$ is the arithmetic graph of some orbit of $S(V_0)$, and the arithmetic graph of every such orbit arises as a component.

It is convenient to introduce the sub-lattice

$$\Lambda = \Lambda_1 \cap \Lambda_2 = \boldsymbol{Z}^{n+1} \cap I(\boldsymbol{Z}^{n+1}). \tag{76}$$

Note that $e_{n+1} \in \Lambda$, so that Λ is always nontrivial. When I has rational entries, as it does for PETs arising from outer billiards on rational polygons, Λ has full rank. In any case, translations by elements of Λ are isomorphisms of $\Gamma(V_0)$. Thus, it perhaps is convenient to consider $\Gamma(V_0)$ as a graph whose vertex set is Λ_1/Λ, a discrete subgroup of the Euclidean manifold $\boldsymbol{R}^{n+1}/\Lambda$. This advantage of this point of view is just that the components of the resulting graph are in bijection with the orbits of $S(V_0)$. That is, each distinct arithmetic graph arises exactly once.

Note that changing the offset vector V_0 will likely produce a different graph. However, in practice, one can get a general sense of what the graph is like just by picking some random V_0 and drawing pictures.

5.3 Linear Projections

Recall that $\Psi : \mathbf{S} \to \widehat{\mathbf{S}}$ is the basic compactifying map. In this section we explain how to recover the orbit O of a point in $\mathbf{S}$ from the arithmetic graph of the orbit $\widehat{O} = \Psi(O)$. For ease of exposition, we restrict our attention to QTCs which are not quasi-rational. For such QTCs, the map Ψ is injective.

Let $\{e_k\}$ be the standard basis vectors in $\boldsymbol{R}^{n+1}$. In this section we prove the following result.

Lemma 5.4 (Projection) *There are vectors $u_1, ..., u_{n+1} \in \mathbf{S}$ so that*

$$I(e_k) \equiv \Psi(u_k) \mod \boldsymbol{Z}^n, \qquad k = 1, ..., n+1, \tag{77}$$

Define $T : \boldsymbol{Z}^{n+1} \to \boldsymbol{R} \times \boldsymbol{R}/\boldsymbol{Z}$ by the rule

$$T(x_1, ..., x_{n+1}) = u_1 x_1 + ... + u_n x_n, \tag{78}$$

with addition in the second coordinate taken mod $\boldsymbol{Z}$. Given an orbit O of $\mathcal{T}^2$, the projection $T(\Gamma_{\widehat{O}})$ agrees with O up to translation.

We will prove this result through a series of smaller lemmas.

Lemma 5.5 *For any vector $v \in \boldsymbol{Z}^{n+1}$, the transformation of $\widehat{\mathbf{S}}$ given by $x \to x + I(v)$ preserves the image $\Psi(\mathbf{S})$.*

Proof: Recall that $\Lambda_1 = \boldsymbol{Z}^{n+1}$ and $\Lambda_2 = I(\Lambda_1)$. We can describe $\Psi(\mathbf{S})$ as follows. There is a 2-plane $\Pi \subset \boldsymbol{R}^{n+1}$ through the point $\Psi(0,0)$ which is tangent to $\Psi(\mathbf{S})$. Then

$$\Psi(\mathbf{S}) = \pi(\Pi + \Lambda_1). \tag{79}$$

That is, we translate Π by all integer vectors and project back into $\mathbf{S}$. The map I preserves Π, and also pointwise fixes a codimension-2 subspace transverse to Π. For this reason, we have

$$I(\Pi + \Lambda_1) = \Pi + \Lambda_1 \tag{80}$$

On the other hand, $I(\Lambda_1) = \Lambda_2$. Therefore

$$I(\Pi + \Lambda_1) = \Pi + \Lambda_2. \tag{81}$$

Combining these equations, we see that

$$\Pi + \Lambda_1 = \Pi + \Lambda_2. \tag{82}$$

That is, the union of the Λ_2 translates of Π coincides with the union of Λ_1 translates of Π. This implies the lemma. ♠

By the preceding result, The transformation $x \to x + I(e_k)$ preserves $\Psi(\mathbf{S})$. Hence, there is some $u_k \in \mathbf{S}$ so that $I(x_k) = \Psi(v)$. This equation takes place in $\widehat{\mathbf{S}}$, which is to say that it takes place mod $\boldsymbol{Z}^{n+1}$. Hence, Equation 77 can always be solved for $k = 1, ..., n+1$. Since Ψ is injective, the solutions are unique for each k. Note that $u_{n+1} = (0,0)$ because $I(e_{n+1}) \in \boldsymbol{Z}^{n+1}$.

Now let $p \in \mathbf{S}$ be a point and let $\widehat{p} = \Psi(p)$. Let $v = v_{\widehat{p}}$ be as in the definition of the arithmetic graph. We can write

$$v = \sum c_k e_k \tag{83}$$

Since $\widehat{\mathbf{S}}$ is an abelian group, we have

$$\widehat{p} + I(v) = \widehat{p} + \sum c_k I(e_k) = \widehat{p} + \sum c_k \Psi(t_k, t_k) = \widehat{p} + \Psi\Big(\sum c_k u_k\Big). \tag{84}$$

The second coordinate of $\sum c_k u_k$ is taken mod $\boldsymbol{Z}$.

It follows from Theorem 1.2, Theorem 1.3, and the group structure of $\widehat{\mathbf{S}}$ that

$$I(v) = [X_1, X_2, I]^2(\widehat{p}) - \widehat{p} = \Psi(\mathcal{T}^2(p)) - \Psi(p) = \Psi(\mathcal{T}^2(p) - p). \tag{85}$$

Since Ψ is injective, we have

$$\mathcal{T}^2(p) - p = \sum c_k u_k. \tag{86}$$

Summing Equation 86 over the whole orbit, we see that $T(\Gamma_{\widehat{O}})$ and O are translates of each other.

Unboundedness Criterion: Let t_k be the first coordinate of u_k. Let

$$T_1(x_1, ..., x_n) = t_1 x_1 + ... + t_n x_n. \tag{87}$$

The second coordinate of u_k always lies in $[-1/2, 1/2]$. Hence, the orbit O is unbounded iff $\pi_1(O)$, the projection onto the first coordinate is unbounded. For this reason, the orbit O is unbounded if and only if $T_1(\Gamma)$ is unbounded.

5.4 Discussion

Another Unboundedness Criterion: One may object that our unboundedness criterion above is rather indirect. For this reason, we state a more direct unboundedness criterion which has nothing (obviously) to do with the arithmetic graph. Given an infinite orbit O of $\mathcal{T}$, we call $\widehat{O} = \Psi(O)$ *transversely infinite* if $\widehat{O}$ intersects infinitely many components of $\Psi(\mathbf{S}) \cap \Pi_1(X_1)$. It follows directly from compactness that O is an unbounded orbit in $\mathbf{S}$ if and only if $\widehat{O}$ is transversely infinite. Thus, Ψ has unbounded orbits if and only if $[X_1, X_1, I]$ has transversely infinite orbits which intersect $\Psi(\mathbf{S})$.

The Graph in Terms of Double Lattice PETs: One can express the arithmetic graph directly in terms of the action of the double lattice PET: As one iterates ϕ, one produces vectors $\lambda_1, \lambda_2', \lambda_3, \lambda_4', ...$ with $\lambda_1, \lambda_3, ... \in \Lambda_1$ and $\lambda_2', \lambda_4', ... \in \Lambda_2$. Let

$$\lambda_{2k} = I(\lambda_{2k}') \in \Lambda_1. \tag{88}$$

The arithmetic graph Γ of the corresponding orbit is such that

$$\Gamma(m+1) - \Gamma(m) = \lambda_{2m}. \tag{89}$$

A Conjectural Identity: Let $\mathcal{T}$ be a length n QTC such that $\mathcal{T}^2$ is finitary. Let $\mathcal{R}_k$ denote the kth rectangle tiling associated to $\mathcal{T}$. We index the rectangles in $\mathcal{R}_k$ by integers, counting left to right. For our purposes, it doesn't matter which rectangle we index as "rectangle 0", but certainly one can make some convenient choice – e.g., the rightmost rectangle that contains the origin.

Given a point $p \in \mathbf{S}$ we produce numbers $a_1, ..., a_{2n}$ as follows.

- The point p lies in rectangle a_1 of $\mathcal{R}_1$.
- The point $p' = S_1R_1(p)$ lies in rectangle a_2 of $\mathcal{R}_2$.
- The point $p'' = S_2R_2(p)$ lies in rectangle a_3 of $\mathcal{R}_3$.

and so on. The two integers a_k and a_{n+k} name rectangles in the same tiling, so the difference $a_{n+k} - a_n$ is independent of how we choose rectangle 0. We define

$$\gamma(p) = (a_{n+1} - a_1, ..., a_{n+n} - a_n) \in \boldsymbol{Z}^n \tag{90}$$

Given an orbit $O = \{p_k\}$ in $\mathbf{S}$ we define Γ_O so that

$$\Gamma_O(m+1) - \Gamma_O(m) = \gamma(p_m). \tag{91}$$

This definition looks suspiciously like the arithmetic graph, though it is defined entirely in terms of the dynamics of the QTC. We can try to compare Γ_O with the arithmetic graph $\Gamma_{\widehat{O}}$ of the orbit $\widehat{O} = \Psi(O)$. This latter object is defined in terms of the dynamics on the compactification.

For QTCs which come from outer billiards on kites, we noticed that Γ_O and $\Gamma_{\widehat{O}}$ always coincide up to translation. We conjecture this always happens, but we haven't extensively tested the conjecture and we don't know a proof.

6 Examples

6.1 Kites

In [**S2**] we studied the kite $K(A)$ having vertices

$$(-1,0), \qquad (0,1) \qquad (0,-1) \qquad (A,0); \qquad A\in(0,1). \tag{92}$$

The QTCs corresponding to $K(A)$ for $A \in (0,1)$, sit inside a 2-parameter family $\mathcal{F}(A_1,A_2)$ of length 4 QTCs whose second iterates are all finitary. We now describe this family. Given two constants $A_1, A_2 \in (0,1)$ we define

$$B=\frac{1+A_1}{1-A_1}, \qquad C=\frac{1+A_2}{1-A_2}, \qquad D=\frac{B^2}{C^2-1}. \tag{93}$$

The corresponding QTC has parameters parameters

$$q=(1,1,0,1) \qquad r=(1,B,D,B); \qquad s=(C,CD,CD,C). \tag{94}$$

The QTC corresponding to a suitably scaled copy of $K(A)$ is obtained by setting $A_1 = A_2 = A$.

It is worth pointing out that these systems are all palindromic, in the sense that

$$S_4\circ R_4\circ S_3\circ R_3\circ S_2\circ R_2\circ S_1\circ R_1 =$$
$$S_1\circ R_2\circ S_2\circ R_3\circ S_3\circ R_4\circ S_4\circ R_1.$$

In case $A_1 = A_2 = A$, this equation is the manifestation of the bilateral symmetry of the kite. In general, we don't have a geometric interpretation of $\mathcal{F}(A_1,A_2)$ in terms of polygons (or anything else).

For the rest of the discussion we stick to the case $A_1 = A_2 = A$. For any $\alpha \in \boldsymbol{R}/2\boldsymbol{Z}$, the discrete set of horizontal lines

$$\boldsymbol{R}^2_\alpha = \bigcup_{n\in\boldsymbol{Z}} \boldsymbol{R}\times\{\alpha+2n\} \tag{95}$$

is invariant under the square of the outer billiards map. The set

$$I_\alpha = A_{1,+}(R^2_\alpha\cap\Sigma_1)\subset \mathbf{S} \tag{96}$$

is invariant under $\mathcal{T}$. Here $A_{1,+}:\Sigma_1\to\mathbf{S}$ is as in Equation 11. The set I_0 is the union of the positive sloped diagonals of the rectangles in the tiling $\mathcal{R}_1$. The set I_α is a horizontal translate of I_0. As we have normalized it, these segments all have slope 1. The thick dark lines in Figure 6.1 are I_1 and the ligher lines are I_0. Figure 5.1 also shows how $K(A)$ sits inside the strip $\mathbf{S}$.

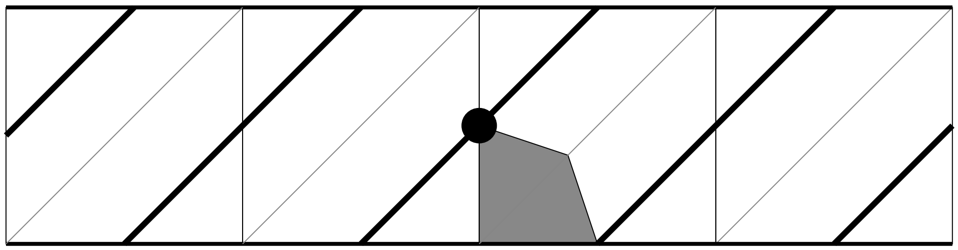

Figure 6.1: The sets I_0 and I_1.

The map $\Psi : \mathbf{S} \to \widehat{\mathbf{S}}$ is given by

$$\Psi(x,y) = \Big(x, \frac{1-a}{1+a}x, \frac{4a}{(1+a)^2}x, \frac{1-a}{1+a}x, y\Big) \mod \boldsymbol{Z}^5. \tag{97}$$

The compactification of $\mathcal{T}$ is typically 4 dimensional. In view of the fact that $r_2 = r_4$, this compactification is contained in

$$(\boldsymbol{R}^5/\boldsymbol{Z}^5) \cap \{x_2 = x_4\}. \tag{98}$$

The image $\Psi(I_\alpha)$ is contained in an invariant subspace given by

$$\Pi_\alpha = \{x_2 = x_4\} \cap \{x_1 - y = \alpha/2\}. \tag{99}$$

The corresponding PET is typically 3-dimensional. In [**S2**, Master Picture Theorem] we described the Π_1 PET in detail.

Now we describe the associated triple (X_1, X_2, I) from Theorem 1.3. The involution I is given by

$$I = \begin{bmatrix} 0 & -1 & \frac{-1-a}{2} & 0 & 0 \\ \frac{a-1}{a+1} & \frac{2a}{1+a} & \frac{a-1}{2} & 0 & 0 \\ \frac{-4a}{(1+a)^2} & \frac{-4a}{(1+a)^2} & \frac{1-a}{1+a} & 0 & 0 \\ \frac{a-1}{a+1} & \frac{a-1}{a+1} & \frac{a-1}{2} & 1 & 0 \\ 1 & 0 & \frac{-1-a}{2} & -1 & -1 \end{bmatrix} \tag{100}$$

The polytope X_j consists of those vectors v such that $M_j(v) \in [-1/2, 1/2]^5$. Here

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{2a}{1+a} & 1 & 0 & 0 & \frac{1-a}{1+a} \\ \frac{-2a}{1+a} & \frac{1-a}{1+a} & 1 & 0 & 1 \\ -1 & 0 & \frac{1+a}{2} & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{101}$$

$$M_2 = \begin{bmatrix} 0 & -1 & \frac{-a-1}{2} & 0 & 0 \\ 0 & 0 & -1 & \frac{a-1}{a+1} & \frac{a-1}{a+1} \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & \frac{-a-1}{2} & -1 & -1 \end{bmatrix} \quad (102)$$

These matrices are such that $I^t M_j = M_{j+1}$, with indices taken mod 2.

Referring to our Unboundedness Criterion, the projection from Equation 87 is given by

$$T_1(x_1, x_2, x_3, x_4) = (2x_1 + 2x_2 + x_3) + Ax_3. \quad (103)$$

Finally, it is worth pointing out that the slightly different projection

$$U_1(x_1, x_2, x_3, x_4) = (x_2 + x_3 + x_4) + Ax_3. \quad (104)$$

yields the arithmetic graph from our monograph [**S2**]. This alternate projection also works in our Unboundedness criterion, at least for orbits of points in Π_1. The pictures associated to U_1 are somewhat prettier than the ones associated to T_1.

Remark: We checked all these formulas with a computer program which illustrates these examples.

6.2 Regular polygons

A calculation shows that the QTC associated to a regular n gon has parameters

$$q = (1, ..., 1), \qquad r = (1, ..., 1), \qquad s = (\alpha_n, ..., \alpha_n), \qquad \alpha_n = 2\cos(\pi/n). \quad (105)$$

In this case, the compactification is also 2 dimensional, and we can identify the compactification with the square torus $\boldsymbol{R}^2/\boldsymbol{Z}^2$. We can think of the above QTC as the nth iterate of the length 1 QTC given by the parameters

$$q = 1, \qquad r = 1, \qquad s = 2\cos(\pi/n). \quad (106)$$

Though not studied directly, this map is essentially equivalent to some maps which have been studied in detail.

- When $n = 3, 6$, this map is basically trivial.

- The case $n = 5$, corresponding to outer billiards on the regular pentagon, has been studied in detail, from a different point of view, by Tabachnikov. See [**T2**].

- The symbolic dynamics of the map for the cases $n = 5, 8$, again considered from a different point of view, has been studied in [**BC**].

- The so-called arithmetic graph of the system, for the case $n = 8$, has been studied in detail in [**S5**].

- The cases $n = 10, 12$ are similar to the cases $n = 5, 8$ in that the map is defined over a quadratic irrational number field. In all such cases, one can say quite a bit.

The remaining cases seem extremely rich, but still are poorly understood. Some experimental work has been done on the case $n = 7$ (by Arek Goetz and Gordon Hughes, separately, for instance) but it seems that there are not yet any definitive rigorous results. A case such as $n = 11$ seems beyond the reach of current technology. The orbit structure seems unbelievably complex.

It probably makes sense to consider the family of maps in Equation 106 to be part of a 1-parameter family of affine PETs where s is allowed to vary freely, say, in the interval $[0, 2]$. We force $s \leq 2$ because this is the limit in Equation 106 as $n \to \infty$.

6.3 QTCs of Length Three

All triangles are equivalent up to affine transformations, so up to affine transformations, there is only one outer billiards system based on a triangle. The corresponding QTC has parameters $q = r = s = (1, 1, 1)$.

On the other hand, there is a 3 parameter family of length 3 QTCs which lead to PETs. An easy exercise in linear algebra shows that the linear part of $\widehat{\mathcal{T}}$ is never the identity when $\mathcal{T}$ is a QTC of length 2. For QTCs of length 3, the parameters

$$(q_1, q_2, q_3) = (0, 0, 0); \quad (r_1, r_2, r_3) = (ab, bc, ca); \quad (s_1, s_2, s_3) = (b^2, c^2, a^2) \tag{107}$$

lead to $\mathcal{T}$ such that $\mathcal{T}^2$ is finitary.

It is worth pointing out that plenty of these QTCs have unbounded orbits. For instance, let $\mathcal{T}$ be the QTC corresponding to $(a,b,c) = (1,2,4)$. Let X be the triangle with vertices

$$(3/2, 1/2), \qquad (2,0), \qquad (2,1/2).$$

An easy calculation shows that

$$\mathcal{T}^6(p) = p + (8,0)$$

for any p in the interior of X. But $\mathcal{T}$ commutes with a horizintal shift of 8 units. Hence

$$\mathcal{T}^{6m}(p) = p + (8m, 0),$$

for any p in the interior of X.

A rational QTC coming from an outer billiards system only has periodic orbits, thanks to the result in [**VS**], [**K**], and [**GS**]. Our example shows that the dynamics of a general QTC can be rather different from the dynamics of a QTC coming from an outer billiards system. Even so, these length 3 QTCs seem to be an attractive family to study.

Now we describe the associated triple (X_1, X_2, I). With the same notation as above, we have

$$I = \begin{bmatrix} 0 & -1 & \frac{(c-b)}{a} & 0 \\ -\frac{c}{b} & 1-\frac{c}{b} & \frac{-bc+c^2}{ab} & 0 \\ -\frac{a}{b} & -\frac{a}{b} & \frac{c}{b} & 0 \\ 1 & \frac{a-b}{c} & -1 & -1 \end{bmatrix} \tag{108}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{a-c}{b} & 1 & 0 & \frac{c}{b} \\ -1 & \frac{b-a}{c} & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{109}$$

$$M_2 = \begin{bmatrix} 0 & -1 & \frac{c-b}{a} & 0 \\ 0 & 0 & -1 & \frac{-c}{b} \\ 0 & 0 & 0 & -1 \\ 1 & \frac{a-b}{c} & -1 & -1 \end{bmatrix} \tag{110}$$

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