

Rectangle Sweepouts and Coincidences

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Abstract

We prove an integral formula for continuous paths of rectangles inscribed in a piecewise smooth loop. We then use this integral formula to show that (with a very mild genericity hypothesis) the number of rectangle coincidences, informally described as the number of inscribed rectangles minus the number of isometry classes of inscribed rectangles, grows linearly with the number of positively oriented extremal chords – a.k.a. diameters – in a polygon.

1 Introduction

A *Jordan loop* is the image of a circle under a continuous injective map into the plane. Toeplitz conjectured in 1911 that every Jordan loop contains 4 points which are the vertices of a square. This is sometimes called the *Square Peg Problem*. For historical details and a long bibliography, we refer the reader to the excellent survey article [M] by B. Matschke, written in 2014, and also Chapter 5 of I. Pak’s online book [P].

Some interesting work on problems related to the Square Peg Problem has been done very recently. The paper of C. Hugelmeyer [H] shows that a smooth Jordan loop always has an inscribed rectangle of aspect ratio $\sqrt{3}$. The paper [AA] proves that any cyclic quadrilateral can (up to similarity) be inscribed in any convex smooth curve. The paper [ACFSST] proves, among other things, that a dense set of points on an embedded loop in space are vertices of a (possibly degenerate) inscribed parallelogram.

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Say that a rectangle R *graces* a Jordan loop γ if the vertices of R lie in γ and if the cyclic ordering on the vertices induced by R coincides with the cyclic ordering induced by γ . Let $G(\gamma)$ denote the space of labeled gracing rectangles. In [S1] we prove the following result.

Theorem 1.1 *Let γ be a Jordan loop. Then $G(\gamma)$ contains a connected set S such that all but at most 4 vertices of γ are vertices of members of S .*

We have a more precise characterization of the possibilities for S in [S1]. We proved Theorem 1.1 by taking a limit of a result for polygons. We now describe this result.

Given a polygon P , we say that a chord d of P is a *diameter* if the two perpendiculars to d based at ∂P do not locally separate ∂P into two arcs. Each diameter can be positively oriented or negatively oriented, but not both. To explain the condition, we rotate the picture so that d is vertical. The endpoints of d divide P into two arcs P_1 and P_2 . Given the non-separating condition associated to a chord, we can say whether P_1 locally lies to the left or right of P_2 in a neighborhood of each endpoint of d . We call d *positively oriented* if the left/right answer is the same at both endpoints. That is, either P_1 locally lies to the left at both endpoints or P_1 locally lies to the right at both endpoints. Figure 1 some examples of positive diameters.

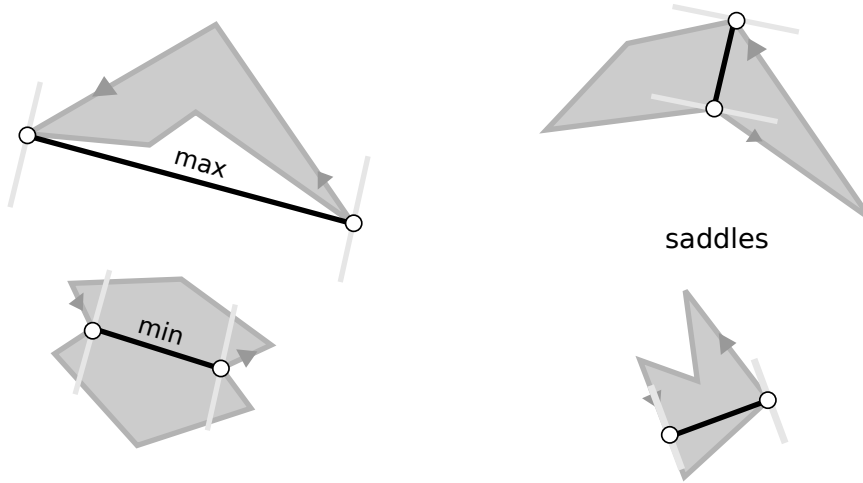


Figure 1: Some positive diameters of polygons.

With respect to the distance function on P , a diameter can be a minimum, a maximum, or neither. We call the third kind *saddles*. Let $\Delta_+(P)$ denote the number of positively oriented diameters of P .

Let Π_N denote the space of embedded N -gons. The set Π_N is naturally an open subset of $(\mathbf{R}^2)^N$ and as such inherits the structure of a smooth manifold. We call a subset $\Pi_N^* \subset \Pi_N$ *fat* if $\Pi_N - \Pi_N^*$ is a finite union of positive codimension submanifolds. In particular, a fat set is open and has full measure.

Theorem 1.2 *There exists a fat subset $\Pi_N^* \subset \Pi_N$ with the following property. For every N -gon $P \in \Pi_N^*$ the space $\Gamma(P)$ is a piecewise-smooth 1-manifold. Each arc component of $\Gamma(P)$ connects two positive diameters of P , and every positive diameter arises as the end of 4 arc components of $\Gamma(P)$. In particular, there are $2\Delta_+(P)$ arc components of $\Gamma(P)$.*

The reason that there are 4 arc components connecting every pair of positive diameters is that we are considering cyclically labeled rectangles. Each of the 4 components is obtained from each other one by cyclically relabeling.

Now we describe the results we prove in this paper. Given a rectangle R , we let $X(R)$ and $Y(R)$ respectively denote the lengths of the first and second sides of R . For any continuous path of rectangles in $\Gamma(P)$ which is either a closed loop or which connects two diameters of P , we define the *shape curve* $Z(\alpha)$. This curve is given by

$$Z(\alpha, t) = (X(R_t), Y(R_t)). \tag{1}$$

Here $t \rightarrow R_t$ is a parametrization of α .

When α is a closed loop, $Z(\alpha)$ is a closed loop as well. When α is an arc component, $Z(\alpha)$ is an arc, not necessarily embedded, that starts and ends on the coordinate axes. Figure 2 shows two of the possibilities.

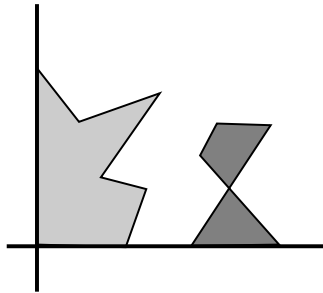


Figure 2: Shape curves associated to hyperbolic and null arcs.

In the first case, one endpoint of α lies on the X -axis and the second endpoint lies on the Y -axis. As in [S1] we call such arcs *hyperbolic arcs*. In

the other cases, both ends lie on the same axis. We call such components *null arcs*. In the arc cases, we augment $Z(\alpha)$ by adjoining the relevant parts of the coordinate axes so as to create a closed loop. We have shaded in the regions bounded by these closed loops. We call this augmented loop the *shape loop* associated to α and give it the same name.

In [S2] we found a kind of integral formula associated to the shape loop, though we stated it in a different context. This invariant is quite similar to the integral invariant in [Ta], though we use it in a different context. (In §2.4 we give a sample result from [S2].) Here we adapt the invariant to the present situation and prove the following theorem.

Theorem 1.3 *Let P be any piecewise smooth Jordan loop. Let α be a piecewise smooth path in $\Gamma(P)$. If α is a hyperbolic arc then the signed area of the region bounded by $Z(\alpha)$ equals (up to sign) the area of the region bounded by P . If α is either a null arc or a closed loop, then the signed area of the region bounded by $Z(\alpha)$ is 0.*

Theorem 1.3 says something about the number of coincidences that appear amongst the inscribed rectangles. We will give an example which explains the connection. Since the shape loop associated to a null component bounds a region of area 0, the shape curve must have a self-intersection. This self-intersection corresponds to a pair of isometric rectangles inscribed in the polygon. Now we formulate a general result. We call two labeled rectangles *really distinct* if their unlabeled versions are also distinct. Thus, two relabelings of the same rectangle are not really distinct.

We define the multiplicity of the pair (X, Y) as follows.

- $\mu(X, Y) = n - 1$ if there are $n > 1$ really distinct labeled rectangles R_1, \dots, R_n inscribed in P such $X(R_j) = X$ and $Y(R_j) = Y$ for all $j = 1, \dots, n$. We also allow $n = \infty$,
- $\mu(X, Y) = 0$ if there are 0 or 1 such rectangles.

We define

$$M(P) = \sum \mu(X, Y), \tag{2}$$

where the sum is taken over all pairs (X, Y) . Typically this is a sum with finitely many finite nonzero terms. There is a more natural (but somewhat informal) way to think about $M(P)$. Suppose that we color all the points in $\Gamma(P)$ according to the isometry class of rectangles they represent. Then $M(P)$ is the number of points minus the number of colors.

Theorem 1.4 For each $P \in \Pi_N^*$ we have $M(P) \geq 2(\Delta_+(P) - 2)$.

When P is an obtuse triangle we have $M(P) = 0$ and $\Delta_+(P) = 2$, so the result is sharp in a trivial way.

Some version of Theorem 1.4 is true for an arbitrary polygon, but here we place a mild constraint so as to make the proof easier. Let P be a polygon. We call a diameter S of P *tricky* if the endpoints of S are vertices of P and if at least one of the edges of P incident to S is perpendicular to S .

Theorem 1.5 If P has no tricky diameters, $M(P) \geq \frac{1}{16}(\Delta_+(P) - 2)$.

The rest of the paper is devoted to proving the results above.

2 The Integral Formula

2.1 The Differential Version

Let J be a piecewise smooth Jordan loop and let R be a labeled rectangle that graces J . For each $j = 1, 2, 3, 4$ we let A_j denote the signed area of the region R_j^* bounded by the segment $\overline{R_j R_{j+1}}$ and the arc of J that connects R_j to R_{j+1} and is between these two points in the counterclockwise order. Figure 3 shows a simple example. The signs are taken so that the signed areas are positive in the convex case, and then in general we define the signs so that the signed areas vary continuously.

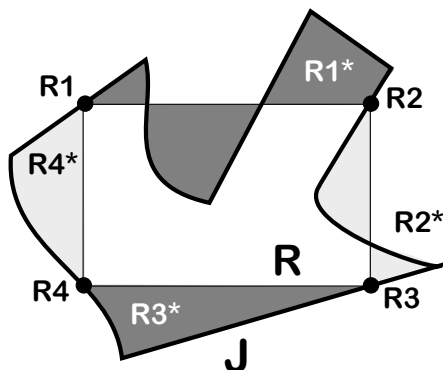


Figure 3: The curve J , the rectangle R and the regions R_j^* for $j = 1, 2, 3, 4$.

Assuming that J is fixed, we introduce the quantity

$$A(R) = (A_1 + A_3) - (A_2 + A_4). \quad (3)$$

We also have the point $(X, Y) \in \mathbf{R}^2$, where

$$X = \text{length}(\overline{R_1 R_2}), \quad Y = \text{length}(\overline{R_2 R_3}), \quad (4)$$

Assuming that we have a piecewise smooth path $t \rightarrow R_t$ of rectangles gracing J , we have the two quantities

$$A_t = A(R_t), \quad (X_t, Y_t) = (X(R_t), Y(R_t)). \quad (5)$$

If t is a point of differentiability, we may take derivatives of all these quantities. Here is the main formula.

$$\frac{dA}{dt} = Y \frac{dX}{dt} - X \frac{dY}{dt}. \quad (6)$$

It suffices to prove this result for $t = 0$. This formula is rotation invariant, so for the purposes of derivation, we rotate the picture so that the first side of R_0 is contained in a horizontal line, as shown in Figures 3 and 4. When we differentiate, we evaluate all derivatives at $t = 0$.

We write

$$\frac{dR_j}{dt} = (V_j, W_j). \quad (7)$$

Up to second order, the region $R_1^*(t)$ is obtained by adding a small quadrilateral with base X_0 and adjacent sides parallel to $t(V_1, W_1)$ and $t(V_2, W_2)$. Up to second order, the area of this quadrilateral is

$$\frac{X(W_1 + W_2)}{2}.$$

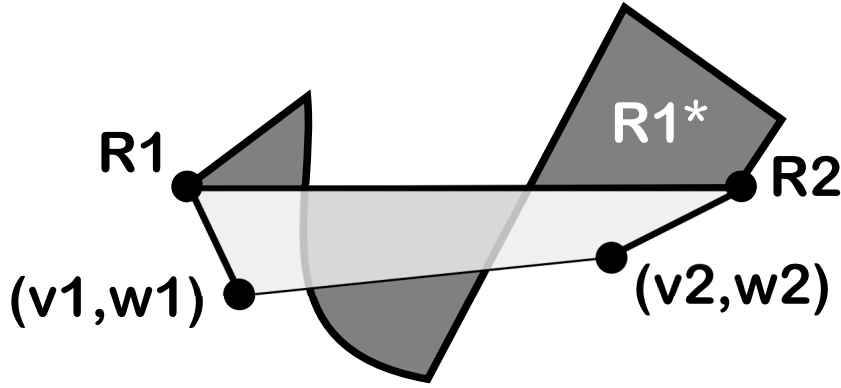


Figure 4: The change in area.

From this equation, we conclude that

$$\frac{dA_1}{dt} = -\frac{X(W_1 + W_2)}{2}. \quad (8)$$

We get the negative sign because the area of the region increases when W_1 and W_2 are negative. A similar derivation gives

$$\frac{dA_3}{dt} = +\frac{X(W_3 + W_4)}{2}. \quad (9)$$

Adding these together gives

$$\frac{dA_1}{dt} + \frac{dA_3}{dt} = X \times \left[\frac{W_3 - W_1}{2} \right] + X \times \left[\frac{W_4 - W_2}{2} \right] =$$

$$-X \times \left[\frac{1}{2} \frac{dY}{dt} \right] + -X \times \left[\frac{1}{2} \frac{dY}{dt} \right] = -X \frac{dY}{dt}. \quad (10)$$

A similar derivation gives

$$\frac{dA_2}{dt} = -\frac{X(V_2 + V_3)}{2}, \quad \frac{dA_4}{dt} = +\frac{X(V_4 + V_1)}{2}. \quad (11)$$

Adding these together gives

$$\frac{dA_2}{dt} + \frac{dA_4}{dt} = -Y \frac{dX}{dt}. \quad (12)$$

Subtracting Equation 12 from Equation 10 gives

$$\frac{dA}{dt} = -X \frac{dY}{dt} + Y \frac{dX}{dt}, \quad (13)$$

as claimed.

2.2 The Integral Version

Let $\omega = -XdY + YdX$. Here we think of ω as a 1-form. Suppose that we have parameterized our curve of rectangles so that the parameter t runs from 0 to 1. Integrating Equation 13 over the piecewise smooth path, we see that

$$A_1 - A_0 = \int_Z \omega. \quad (14)$$

Here Z is the shape curve associated to the path of rectangles. We can interpret this integral geometrically. Letting $O = (0, 0)$, consider the closed loop

$$Z' = \overline{O, Z_0} \cup Z \cup \overline{Z_1, O}. \quad (15)$$

Since ω vanishes on vectors of the form (h, h) , we see that

$$A_1 - A_0 = \int_Z \omega = \int_{Z'} \omega = - \int \int_{\Omega} 2dxdy = -2 \text{ area}(\Omega). \quad (16)$$

Here Ω is the region bounded by Z' . The last line of the equation refers to the signed area of Ω .

Proof of Theorem 1.3: Suppose first that α is a piecewise smooth loop rectangles which grace the Jordan curve J . Then the curve Z is already a

closed loop, and the signed area of the region bounded by Z is the same as the signed area bounded by Z' . Since $A_1 = A_0$ in this case, we see that Z bounds a region of signed area 0.

If α is a null arc, then R_0 and R_1 both have the same aspect ratio, either 0 or ∞ . In either case, we have $A_0 = A_1$. The common value is, up to sign, the area of the region bounded by J . In this case, Z starts and stops on one of the coordinate axes, and the region bounded by Z has the same area as the shape loop, $Z \cup \overline{Z_0 Z_1}$. So, in this case we also see that the shape loop bounds a region of area 0.

If α is a hyperbolic arc, then $A_0 = -A_1$ and both quantities up to sign equal the area of the region bounded by J . At the same time Z' is precisely the shape loop in this case. So, we see that twice the area of the region bounded by J equals twice the area of the region bounded by Z , up to sign. Cancelling the factor of 2 gives the desired result. ♠

2.3 Generic Coincidences

In this section we prove Theorem 1.4. Suppose that P is an N -gon that satisfies the conclusions of Theorem 1.2. This happens if $P \in \Pi_N^*$, but it might happen more generally. In any case, the space $\Gamma(P)$ of gracing rectangles has $2\Delta(P)$ arc components. There is a $\mathbf{Z}/4$ action on $\Gamma(P)$ and this action freely permutes the arc components of $\Gamma(P)$.

We let $\delta = \Delta/2$ and we let $\alpha_1, \dots, \alpha_\delta$ denote a complete set of representatives of these arc components modulo the $\mathbf{Z}/4$ action. It suffices to show that the sum in Equation 2 is at least $\delta - 1$ when we restrict our attention to the components just listed.

Consider those arcs on our list which are null arcs. The shape loops associated to each of these arcs bound regions of area 0 and hence the corresponding loop has a double point. Each double point corresponds to a distinct pair that adds 1 to the total count for $M(J)$. The remaining rectangle coincidences involve rectangles not associated to these arcs or to their images under the $\mathbf{Z}/4$ action.

Now consider those arcs on our list which are hyperbolic arcs whose shape loops are not embedded. In exactly the same way as above, each of these arcs contributes 1 to the count for $M(J)$ and the rectangle pairs involved are distinct from the ones we have already considered. Again, the remaining rectangle coincidences involve rectangles not associated to these arcs or to

their images under the $\mathbf{Z}/4$ action.

Remark: Before we move on to the last case, we mention that the count above might be an under-approximation, even in case there is just one double point per shape loop considered. Consider the simple situation where there are just 2 null arcs. It might happen that the rectangle pairs corresponding to these 2 arcs are congruent to each other. This would give us a 4 congruent gracing rectangles and would contribute 3 rather than 2 to the total count.

Finally, consider the d hyperbolic arcs on our list which have embedded shape loops. If α_1 and α_2 are two such arcs, then $Z(\alpha_1)$ and $Z(\alpha_2)$ are two closed loops which bound the same area. If these loops did not intersect in the positive quadrant, then either the region bounded by $Z(\alpha_1)$ would strictly contain the region bounded by $Z(\alpha_2)$ or the reverse. This contradicts the fact that these two regions have the same area. Hence $Z(\alpha_1)$ and $Z(\alpha_2)$ intersect in the positive quadrant, and the intersection point corresponds to a coincidence involving a rectangle associated to α_1 and a rectangle associated to α_2 . Call this the *intersection property*.

We label so that $\alpha_1, \dots, \alpha_d$ are the hyperbolic arcs having embedded shape loops. We argue by induction that these d arcs contribute at least $d - 1$ to the count for $M(J)$. If $d = 1$ then there is nothing to prove. By induction, rectangle coincidences associated to the arcs $\alpha_1, \dots, \alpha_{d-1}$ contribute $d - 2$ to the count for $M(J)$.

By the intersection property, α_d intersects each of the other arcs, and $\Gamma(J)$ is a manifold, there is at least one new rectangle involved in our count, namely one that corresponds to a point on $Z(\alpha_d)$ that is also on some of the shape loop. The corresponding rectangle adds 1 to the count in Equation 2, one way or another. So, all in all, we add $d - 1$ to the count for $M(J)$ by considering the rectangle coincidences associated to $\alpha_1, \dots, \alpha_d$. This proves what we want.

2.4 A Non-Squeezing Result

Here we explain how the invariant above implies one of our main results in [S2]. Really, it is the same proof. The material in this section plays no role in the rest of the paper.

Suppose that γ_1 and γ_2 are 2 piecewise smooth curves which are disjoint. Suppose also that at each end, γ_j coincides with a line segment. Finally

suppose that these line segments are parallel at each end, so to speak. Figure 5 shows what we mean.

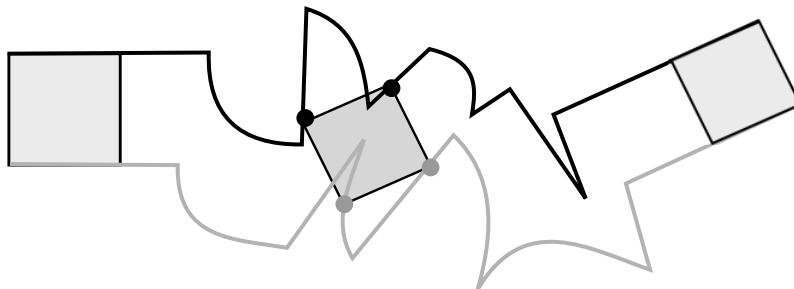


Figure 5: Sliding a square along a track.

Suppose that we have a piecewise smooth family of rectangles, all having the same aspect ratio, that starts at one end, finishes at the other, and remains inscribed in $\gamma_1 \cup \gamma_2$ the whole time. We imagine $\gamma_1 \cup \gamma_2$ as being a kind of track that the rectangle slides along (changing its size and orientation along the way). Figure 5 shows an example in which case the rectangle is a square. In Figure 5 we show the starting rectangle R_0 , the ending rectangle R_1 , and some R_t for $t \in (0, 1)$. This is just a hypothetical example.

We can complete the union $\gamma_1 \cup \gamma_2$ to a piecewise smooth Jordan loop by extending the ends of one or both of these curves, if necessary, and then dropping perpendiculars. Let Ω be the region bounded by this loop. The shape curve associated to our path lies on a line through the origin, and our 1-form ω vanishes on such lines. Referring to the invariant above, we therefore have $A(R_0) = A(R_1)$. But, after suitably labeling the rectangles in our family, we have

$$A(R_j) = \text{area}(\Omega) - \text{area}(R_j).$$

Hence R_0 and R_1 have the same area. Since they also have the same aspect ratio, they have the same side-lengths. This is to say that the perpendicular distance between the end of γ_1 and the end of γ_2 is the same at either end. This is a kind of non-squeezing result.

In particular, our result shows that Figure 5 depicts an impossible situation. There is no way to slide a square continuously through the shown “track” because the widths are different at the 2 ends.

3 The General Case

3.1 Rectangles Inscribed in Lines

The goal of this chapter is to prove Theorem 1.5. We plan to take a limit of the result in Theorem 1.4.

Let $E = (E_1, E_2, E_3, E_4)$ be a collection of 4 line segments, not necessarily distinct. We say that a rectangle R *graces* E if the vertices R_1, R_2, R_3, R_4 of R go in cyclic order, either clockwise or counterclockwise, and $R_i \in E_i$ for all $i = 1, 2, 3, 4$. We allow R to be degenerate. Let $\Gamma(E) \subset (\mathbf{R}^2)^4$ denote the set of rectangles gracing E . Note

We call a point $p \in \Gamma(E)$ *degenerate* if every neighborhood of p in Γ_E contains points corresponding to infinitely many distinct but isometric rectangles. We call E *degenerate* if there is some $p \in \Gamma(E)$ which is degenerate.

Lemma 3.1 *Suppose that E is nondegenerate. $\Gamma(E)$ is the intersection of a conic section with a rectangular solid.*

Proof: Let $E = (E_1, E_2, E_3, E_4)$ be a 4-tuple of lines. We rotate so that none of the segments is vertical, so that we may parameterize the lines containing our segments by their first coordinates. Let L_j be the line extending E_j . We identify \mathbf{R}^3 with triples (x_1, x_2, x_3) where $p_j = (x_j, y_j) \in L_j$. We let p_4 be such that $p_1 + p_3 = p_2 + p_4$. In other words, we choose p_4 so that (p_1, p_2, p_3, p_4) is a parallelogram.

Let $\Gamma(L)$ denote the set of rectangles gracing L . We describe the subset $\Gamma'(L) \subset \mathbf{R}^3$ corresponding to $\Gamma(L)$. The actual set $\Gamma(L)$ is the image of $\Gamma'(L)$ under a linear map from \mathbf{R}^3 into $(\mathbf{R}^2)^4$.

The condition that $p_4 \in L_4$ is a linear condition. Therefore, the set $(x_1, x_2, x_3) \in \mathbf{R}^3$ corresponding to parallelograms inscribed in L is a hyperplane Π . The condition that our parallelogram is a rectangle is $(p_3 - p_2) \cdot (p_1 - p_2) = 0$. This condition defines a quadric hypersurface H in \mathbf{R}^3 . The intersection $\Gamma'(L) = \Pi \cap H$ corresponds to the inscribed rectangles.

$\Pi \cap H$ is either a plane or a conic section. In the former case, E is degenerate. In the latter case, every point $\Pi \cap H$ is either an analytic curve or two crossing lines. Since $\Gamma(L)$ is the image of $\Gamma'(L)$ under a linear map, the set $\Gamma(L)$ is also a conic section.

Let $[E] = E_1 \times E_2 \times E_3 \times E_4$. The $[E]$ is a rectangular solid. We have $\Gamma(E) = \Gamma(L) \cap [E]$. ♠

Lemma 3.2 *When E is non-degenerate, $\Gamma(E)$ has at most $64 = 2^8$ connected components.*

Proof: We use the notation from the previous lemma. Note $[E]$ is bounded by 8 hyperplanes and a conic section either lies in a hyperplane or intersects it at most twice. So, each boundary component of $[E]$ cuts $\Gamma(L)$ into at most 2 components. ♠

We call a polygon P *degenerate* if some 4-tuple of edges associated to P is degenerate. Otherwise we call P *non-degenerate*.

Lemma 3.3 *Let P be a non-degenerate polygon. The space $\Gamma(E)$ is a graph having analytic edges and degree at most 32.*

Proof: Each rectangle R can grace at most 16 different 4-tuples of edges of P , because each vertex can lie in at most 2 segments. Hence, each $p \in \Gamma(P)$ lies in the intersection of at most 16 distinct $\Gamma(E)$. Since $\Gamma(E)$ is the intersection of a conic section with a rectangular solid, $\Gamma(E)$ is a graph with analytic edges and maximum degree 4. From what we have said above, $\Gamma(P)$ is a graph with analytic edges and maximum degree $64 = 16 \times 4$.

We can cut down by a factor of 2 as follows. The only time a point of $\Gamma(P)$ lies in more than 8 spaces $\Gamma(E)$ is when p corresponds to a gracing rectangle whose every vertex is a vertex of P . In this case, p is a vertex of $[E]$ for each 4-tuple E that the rectangle graces. But then p has degree at most 2 in each $\Gamma(E)$. So, this exceptional case produces vertices of degree at most 32. ♠

3.2 The Inscribing Sequence

A generic polygon P satisfies the conclusions of Theorem 1.4. For such polygons, any 4-tuple which supports a gracing rectangle is nice.

We label the sides of P by $\{1, \dots, N\}$. Let Ω denote the set of ordered 4-element subsets of $\{1, \dots, N\}$, not necessarily distinct. Consider some embedded arc $\alpha \subset \Gamma(P)$ of inscribed rectangles. α defines a finite sequence Σ of elements of Ω . We simply note which edges of P_n contain any given rectangle and then we order the elements of Ω we get. We call Σ the *inscribing sequence* for α .

Lemma 3.4 Σ has length at most κN^4 .

Proof: If Σ had length longer than this, then we could find a single 4-tuple E of edges such that the subset of α supported by E has at least 82 components. In other words the sequence would have to return to the 4-tuple describing E at least 82 times. The arcs of $\Gamma(E)$ corresponding to these returns are disconnected from each other, because otherwise α would be a loop rather than an arc. This contradiction proves our claim. ♠

3.3 Stable Diameters

For the rest of the chapter, we use the word *diameter* to mean a positively oriented diameter, in the sense discussed in the introduction.

Let P be a polygon and let S be a diameter of P . We call S *stable* if

- At least one endpoint of S is a vertex of P .
- If v is an endpoint of S and e is an edge of P incident to P at v , then S and e are not perpendicular.

Lemma 3.5 Suppose that P has no tricky diameters. If P has an unstable diameter, then P is non-degenerate.

Proof: This is a case-by-case analysis. Suppose first that P has a diameter S whose endpoints are not vertices of P . Then the endpoints of S lie in the interior of a pair of parallel edges of P . But then P is degenerate. Suppose that P has a diameter S having one endpoint which is a vertex v of P . The other endpoint of S lies in the interior of an edge e' of P . By definition e' and S are perpendicular. If S is not stable, then one of the edges e of P is perpendicular to S and hence parallel to e' . But then we can shift S over a bit and produce a diameter of P whose endpoints lie in the interior of e and e' . Again, P is degenerate. The remaining unstable diameters are (in the technical sense) tricky. ♠

In view of the preceding result, it suffices to prove Theorem 1.5 under the assumption that P is non-degenerate and has all stable diameters.

3.4 Limits of Diameters

Let P be an N -gon with stable diameters. We can find a sequence $\{P_n\}$ of generic N -gons converging to P . Each P_n satisfies the conclusions of Theorem 1.4.

Lemma 3.6 *Let D be a diameter of P . The polygon P_n has a diameter D_n such that $\{D_n\}$ converges to D .*

Proof: Since P only has stable diameters, there are just 2 cases to consider. Suppose first that D connects two vertices v and w of P . The polygon P_n has vertices v_n and w_n which converge respectively to v and w as $n \rightarrow \infty$. Let D_n be the chord whose endpoints are v_n and w_n . By construction, D_n converges to D and for large n this chord is a diameter.

Suppose now that D connects a vertex v to a point in the interior of an edge e . Let v_n and e_n be the corresponding vertex and edge of P_n . Since $v_n \rightarrow v$ and since $e_n \rightarrow e$ we see that eventually there is a chord D_n that has v_n as one endpoint and has the other endpoint perpendicular to e_n . By construction $D_n \rightarrow D$ and eventually D_n is a diameter of P_n . ♠

Lemma 3.7 *If $\{D_n\}$ is a sequence of diameters of P_n , then $\{D_n\}$ converges on a subsequence to a diameter of P .*

Proof: Given the sequence $\{D_n\}$ we can pass to a subsequence so that the endpoints of these diameters converge. The limiting segment D , provided that it has nonzero length, must be a diameter of P because the required condition is a closed condition. We just have to see that the length of $\{D_n\}$ does not shrink to 0. Note that D_n is at least as long as the shortest diameter of P_n . Furthermore, there is a positive lower bound to the length of any edge of P_n , independent of n . So, if the length of D_n converges to 0, there are two non-adjacent vertices of D_n whose distance converges to 0. This contradicts the fact that $\{P_n\}$ converges to the embedded polygon P . ♠

We think of a diameter as a subset of $(\mathbf{R}^2)^2$, and in this way we can talk about the distance between two diameters of P_n .

Lemma 3.8 *Suppose that $\{D_n\}$ and $\{D'_n\}$ are two sequences of diameters such that the distance from D_n to D'_n converges to 0 as $n \rightarrow \infty$. Then $D_n = D'_n$ for n sufficiently large.*

Proof: Let v_n and w_n be the endpoints of D_n and let v'_n and w'_n be the endpoints of D'_n . We label so that $\|v_n - v'_n\|$ and $\|w_n - w'_n\|$ both tend to 0. In all cases, we can re-order so that v_n is a vertex of P_n and v'_n is not. In other words, v'_n lies in the interior of an edge e'_n of P_n . Since v'_n converges to v_n , a vertex of P_n , the segment e'_n becomes perpendicular to D'_n in the limit. This contradicts the fact that P has only stable diameters. ♠

Corollary 3.9 *For n sufficiently large, there is a bijection between the diameters of P_n and the diameters of P such that each diameter of P is matched with a sequence of diameters of P_n which converges to P .*

Proof: This is an immediate consequence of the preceding 3 lemmas. ♠

We truncate our sequence of polygons so that the last corollary holds for all n . For each n , these diameters are paired together by the arc components of the manifold $\Gamma(P_n)$. We pass to a further subsequence so that the same pairs arise for each n . This gives us a well defined way to pair the diameters of D . We say that two diameters of D are *partners* if and only if the corresponding diameters of D_n are paired together.

Lemma 3.10 *Each pair of partner diameters in P is connected by a piecewise smooth path in $J(P)$.*

Proof: Let A and B be two partner diameters of P . Let A_n and B_n be the corresponding diameters of P_n . Let α_n be the arc in $\Gamma(P_n)$ which connects A_n and B_n . To understand the convergence of $\{\alpha_n\}$ we work in the Hausdorff topology on the set of compact subsets of $(\mathbf{R}^2)^4$. This ambient space contains $\Gamma(J)$ for any Jordan loop.

We consistently label the sides of P_n and P . Let Σ_n be the inscribing sequence of α_n . By Lemma 3.4 there is a uniform upper bound of κN^4 on the length of Σ_n . Therefore, we may pass to a subsequence so that the inscribing sequence associated to α_n is independent of n . We write

$$\alpha_n = \alpha_{n1}, \dots, \alpha_{nk},$$

where α_{nj} is the arc of rectangles corresponding to the j th element of the sequence in Ω . Here k is the length of the inscribing sequence.

We pass to a subsequence so that $\{\alpha_{n_j}\}$ converges in the Hausdorff topology to a subset $\alpha_j \subset \alpha$. The set α_j is connected and contained in a subset of $\Gamma(E)$, where E is the 4-tuple of edges corresponding to the j th element of Ω . From the discussion in §3.1, we see that α_j is a compact, connected algebraic arc. By construction α_j and α_{j+1} share one point common for all j . This vertex is the limit of the sequence $\{\alpha_{n_j} \cap \alpha_{n_{j+1}}\}$.

The description above reveals α to be a piecewise smooth arc connecting the two diameters A and B . ♠

3.5 The End of the Proof

Let P be a polygon. We still assume that P has stable diameters, so that the results from the previous section apply. We know from Lemma 3.10 that the diameters of P are paired in some way, and each pair is connected by some piecewise smooth path of gracing rectangles. We can erase any loops that these paths have and thereby assume that all these paths are embedded. Next, we can assume that every 2 arcs in the collection intersect each other in at most one point. Otherwise, we can do a splicing operation to decrease the number of intersection points. (See Figure 6 below.) The splicing operation may change the way that the diameters are paired up, but this doesn't bother us. Finally, we can make our choice of connectors invariant under the $\mathbf{Z}/4$ re-labelling action.

As in the proof of Theorem 1.4 we let $\delta = \Delta_+(P)/2$ and we chose a collection $\alpha_1, \dots, \alpha_\delta$ of connecting arcs which has one representative in each orbit of the $\mathbf{Z}/4$ action.

Suppose that our collection of paths contains two hyperbolic arcs α_1 and α_2 that intersect. Each path connects a (degenerate) rectangle of aspect ratio 0 to a (degenerate) rectangle of aspect ratio ∞ . By splicing the paths together and then re-dividing them, we produce 2 new paths β_1 and β_2 such that each β_j connects two degenerate rectangles of the same aspect ratio. In other words, we can do a cut-and-paste operation at an intersection point to replace the two hyperbolic arcs by null arcs. If necessary, we can erase any loops created in this process. Figure 6 shows this operation.

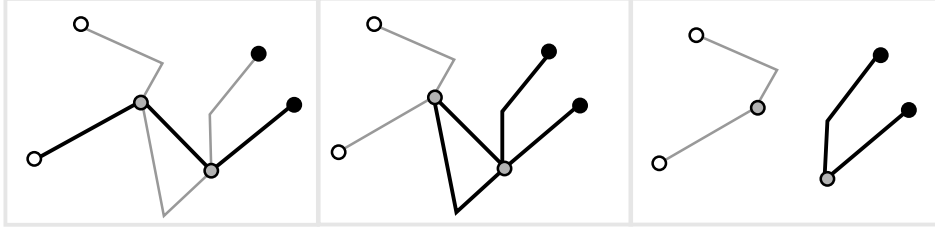


Figure 6: The splicing operation.

Suppose first that there are $\delta/2$ arcs in our collection that are hyperbolic arcs. Then this collection is an embedded 1-manifold contained in $\Gamma(P)$. Just using these arcs, the same argument as in the proof of Theorem 1.4 shows that

$$M(P) \geq \Delta_+(P) - 2.$$

That is, we get the same answer as in Theorem 1.4 except for the factor of $1/2$.

Now suppose that there are at least $\delta/2$ null arcs. For the rest of the proof we just deal with these null arcs. Let $\Gamma_1(P)$ denote the union of these null arcs. We know that $\Gamma_1(P)$ is a subset of $\Gamma(P)$ and also a graph with algebraic edges and maximum valence at most 32. Let $\widehat{\Gamma}_1$ denote the formal disjoint union of these embedded null arcs. The space $\widehat{\Gamma}_1$ is a 1-manifold, just a union of arcs, and the “forgetful map” $\phi : \widehat{\Gamma}_1 \rightarrow \Gamma_1$ is at most 16 to 1.

The same argument as in the proof of Theorem 2.3 says that there are δ distinct points in $\widehat{\Gamma}_1$, two per arc, corresponding to rectangle coincidences. Let S be the set of these points. The image $\phi(S)$ contains at least $\delta/16$ points. For each of these points, there is a second point corresponding to an isometric rectangle. We know this because the map ϕ is injective on each null arc, and each null arc contains 2 points of S . So, we can match our $\delta/16$ points into $\delta/32$ distinct pairs of points, corresponding to pairs of isometric but distinct rectangles in $\Gamma(P)$. This adds a count of $\delta/32$ to $M(P)$. To make the comparison with Theorem 1.4 cleaner, we work with $(\delta - 1)/32$ instead.

In the case at hand, we get the same bound as in Theorem 1.4 except for the factor of $1/32$. Going back to the count of labeled rectangles, we have

$$M(P) \geq \frac{1}{16}(\Delta_+(P) - 2).$$

This completes the proof of Theorem 1.5.

4 References

- [AA] A. Akopyan and S Avvakumov, *Any cyclic quadrilateral can be inscribed in any closed convex smooth curve*. arXiv: 1712.10205v1 (2017)
- [ACFSST] J. Aslam, S. Chen, F. Frick, S. Saloff-Coste, L. Setiabrate, H. Thomas, *Splitting Loops and necklaces: Variants of the Square Peg Problem*, arXiv 1806.02484 (2018)
- [CH] D. Hilbert and S. Cohn-Vossen, *Geometry and The Imagination*, Chelsea Publishing Company (American Math Society), 1990
- [H] C. Hugelmeyer, *Every Smooth Jordan Curve has an inscribed rectangle with aspect ratio equal to $\sqrt{3}$* . arXiv 1803:07417 (2018)
- [M] B. Matschke, *A survey on the Square Peg Problem*, Notices of the A.M.S. Vol 61.4, April 2014, pp 346-351.
- [S1] R. E. Schwartz, *A Trichotomy for Rectangles Inscribed in Jordan Loops*, preprint, 2018
- [S2] R. E. Schwartz, *Four lines and a Rectangle*, preprint, 2018
- [Ta], T. Tao, *An integration approach to the Toeplitz square peg conjecture* Forum of Mathematics, Sigma, 5 (2017)
- [W] S. Wolfram, *The Mathematica Book*, 4th ed. Wolfram Media/Cambridge University Press, Champaign/Cambridge (1999)