Four Lines and a Rectangle

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1 Introduction

While studying properties of rectangles inscribed in Jordan curves [**S**], I discovered some configuration theorems about rectangles inscribed in 4-tuples of lines. (See the references for some articles related to inscribing polygons in curves.) The purpose of this paper is to present those theorems and also an application to the simplest case of a conjecture in [**S**]. The proofs I currently have are somewhat more computational than I would like, but occasionally a geometric idea makes an appearance.

Let $L = (L_1, L_2, L_3, L_4)$ be a 4-tuple of lines in $\boldsymbol{R}^2$. We call L *nice* if

- The 4 lines of L do not all contain the same point.
- No two lines of L are parallel.
- Some line of L is not perpendicular to any other line of L.

The generic configuration is nice. We work with nice configurations to make the results as clean as possible.

We say that a rectangle R is *inscribed in* L if the vertices (R_1, R_2, R_3, R_4) go cyclically around R (either clockwise or counterclockwise) and satisfy $R_i \in L_i$ for $i = 1, 2, 3, 4$. We allow the degenerate cases where $R_1 = R_2$ or $R_2 = R_3$. We call these degenerate rectangles *diagonals* of L. One diagonal connects $L_1 \cap L_2$ to $L_3 \cap L_4$ and the other connects $L_2 \cap L_3$ to $L_4 \cap L_1$. We call L *singular* if the diagonals of L lie in orthogonal lines, and otherwise *regular*. Let $C(L)$ denote the set of centers of rectangles inscribed in L.

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Theorem 1.1 *If L is regular then $C(L)$ is a non-degenerate hyperbola. If L is singular, $C(L)$ is a union of 2 crossing lines.*

Here is more information about the regular case. We define the *slope* of R to be the slope of the first side:

$$\sigma(R) = \mathrm{slope}(\overline{R_1R_2}). \tag{1}$$

We define the *aspect ratio* of R to be the signed ratio of its side lengths:

$$\alpha(R) = \pm\frac{|R_3 - R_2|}{|R_2 - R_1|}. \tag{2}$$

The sign is $+1$ if R is clockwise ordered and -1 if R is counterclockwise ordered. For the diagonals, the aspect ratio is 0 or ∞ and there is no sign. We define the *cross ratio* of L to be the cross ratio of its slopes:

$$\chi(L) = \frac{(m_2 - m_3)(m_4 - m_1)}{(m_1 - m_2)(m_3 - m_4)}, \qquad m_i = \mathrm{slope}(L_i). \tag{3}$$

Note that $\chi(L) = \chi(L')$ if L and L' are affinely equivalent.

Theorem 1.2 *If L is regular, there are 2 unequal values $\alpha_{-1}, \alpha_{+1} \in \boldsymbol{R}$ such that a (unique) rectangle of aspect ratio $\alpha \in \boldsymbol{R} \cup \infty$ is inscribed in L if and only if $\alpha \not\in \{\alpha_{-1}, \alpha_{+1}\}$. There are are 2 unequal values $\sigma_{-1}, \sigma_{+1} \in \boldsymbol{R} \cup \infty$ such that a (unique) rectangle of slope $\sigma \in \boldsymbol{R} \cup \infty$ is inscribed in L if and only if $\sigma \not\in \{\sigma_{-1}, \sigma_{+1}\}$. Furthermore, $\alpha_{-1}\alpha_{+1} = -\chi(L)$ and $\sigma_{-1}\sigma_{+1} = -1$.*

We can interpret $C(L)$ as the set of finite points of an algebraic curve – a conic section – in the projective plane $\boldsymbol{RP}^2$. The map A which sends α to the center of the rectangle of aspect ratio α inscribed in L is a rational diffeomorphism whose domain is $\boldsymbol{R} \cup \infty$ and whose range is the conic. The points $A(\alpha_1)$ and $A(\alpha_2)$ are the infinite points of the conic. The same goes for the map S which sends σ to the center of the rectangle of slope σ inscribed in L. The points $S(\sigma_1)$ and $S(\sigma_2)$ are the infinite points of the conic. We have $A = S \circ \mu$, where μ is the Mobius transformation from Equation 19 below. We have $\mu(\alpha_j) = \sigma_j$ for $j = -1, 0, +1, \infty$. Here σ_0, σ_∞ are the slopes of the two diagonals of L and α_0, α_∞ are their aspect ratios. In particular, the cross ratio of the aspect ratios of 4 rectangles incribed in L equals the cross ratio of the corresponding slopes.

Theorem 1.2 has a geometric meaning. The condition $\sigma_{-1}\sigma_{+1} = -1$ means that the corresponding directions are perpendicular. Thus, in terms of the rotation-invariant metric on the line at infinity in $\boldsymbol{RP}^2$, the rectangles rotate exactly half of the way around when their centers traverse one component of $C(L)$. This has a nice geometric corollary. We call two rectangles *perpendicular* if their corresponding sides are perpendicular.

Corollary 1.3 *Suppose R and R' are perpendicular rectangles inscribed in the same regular configuration L. Then the centers of R and R' are contained in different components of $C(L)$. Hence there is no continuous and bounded path of rectangles inscribed in L which connects R to R'.*

Proof: On $\boldsymbol{R}\cup\infty$ the set $\{\sigma_{+1}, \sigma_{-1}\}$ separates the set $\{\sigma(R), \sigma(R')\}$. Hence, any path of inscribed rectangles connecting R to R' must be unbounded. ♠

The following result adds some perspective to Corollary 1.3, because it says, in particular, that it is false for every singular configuration.

Theorem 1.4 *Suppose L is singular. Then one line of $C(L)$ consists of centers of inscribed rectangles all having the same aspect ratio. Every value $\sigma \in \boldsymbol{R} \cup \infty$ but one arises as the slope of such a rectangle. The other line of $C(L)$ consists of centers of inscribed rectangles all having the same slope. Every $\alpha \in \boldsymbol{R} \cup \infty$ but one arises as the aspect ratio of such a rectangle.*

Theorem 1.4 can be restated as a configuration theorem.

Corollary 1.5 *Suppose that R and R' either have the same slope or the same aspect ratio and are both inscribed in a nice configuration L. Then there are infinitely many rectangles of the same aspect ratio, and infinitely many rectangles of the same slope, inscribed in L. Moreover a continuous path of rectangles inscribed in L, either constant slope or constant aspect ratio, connects R to R'.*

Proof: If the configuration L is regular then by Theorem 1.2 there cannot be two inscribed rectangles having the same aspect ratio or slope. Hence L is singular. This result now follows from Theorem 1.4. ♠

Theorem 1.2 has an application to the study of rectangles inscribed in polygons. Note that a rectangle inscribed in a polygon might have more than one vertex on the same side of the polygon, so the setup is a bit different in this context. Given a polygon P, we say that a chord d of P is a *diameter* if the two perpendiculars to d based at ∂P do not locally separate ∂P into two arcs. With respect to the distance function on P, a diameter can be a minimum, a maximum, or neither. We call the third kind *saddles*.

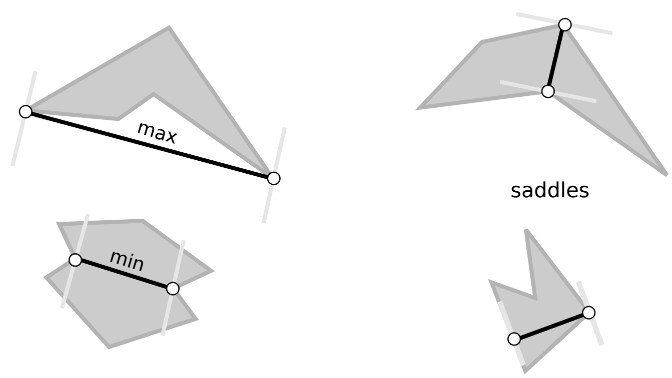

Figure 1: Some diameters of polygons.

A generic polygon P has an even number of diameters, half of which are saddles. We proved in [**S**] that for P generic, these diameters are paired in such a way that each pair is connected by a continuous path of inscribed rectangles. We conjectured that a saddle is always paired with a max or a min. That is, two extreme diameters cannot be paired together. This conjecture is only true generically. For instance, the two diagonals are paired together in a quadrilateral if those diagonals are perpendicular to each other. We use Theorem 1.2 to prove the simplest case of the conjecture.

Theorem 1.6 *Let Q be a quadrilateral such that no two sides of Q are parallel or perpendicular and the diagonals of Q are not perpendicular. There does not exist a continuous path of rectangles inscribed in Q which connects two extreme diameters of Q.*

This paper is organized as follows. In §2 we prove some preliminary results and, most importantly, introduce the expressions D and Δ which arise repeatedly in our proofs. In §3 we prove Theorems 1.1, 1.2, and 1.4. At the end of §3 we prove a sharp version of Corollary 1.3. In §4 we deduce Theorem 1.6 from Theorem 1.2. We mention that we use Mathematica [**W**] for most of the calculations.

2 Preliminaries

2.1 Projective Geometry and Conic Sections

Here we recall a few basic facts about projective geometry and conic sections. See [**CH**] for general background on the subject.

Let $\boldsymbol{RP}^2$ denote the real projective plane, i.e., the space of lines through the origin in $\boldsymbol{R}^3$. One can also think of $\boldsymbol{RP}^2$ as the space of equivalence classes of nonzero vectors in $\boldsymbol{R}^3$, with two vectors being equivalent if they are scalar multiples of each other. The coordinates $[x : y : z]$ denote such an equivalence class. We identify $\boldsymbol{R}^2 \subset \boldsymbol{RP}^2$ with the *affine patch*

$$\{[x : y : 1] |\ x, y \in \boldsymbol{R}^2\}.$$

The *projective transformations* are diffeomorphisms of $\boldsymbol{RP}^2$ induced by the action of invertible 3×3 real matrices.

A *nondegenerate conic section* is the solution set of an irreducible homogeneous polynomial of degree 2, considered as a subset of $\boldsymbol{RP}^2$. A conic section intersect the affine patch in an ellipse, a hyperbola, or a parabola. Projective transformations transitively permute the nondegenerate conic sections. One beautiful thing about projective geometry is that there is just one nonsingular conic section up to projective transformations.

Given a 3×3 invertible matrix $A = A_{ij}$, we introduce the parametric curve

$$\Gamma_A(\alpha) = [A_{00} + A_{01}\alpha + A_{02}\alpha^2 : A_{10} + A_{11}\alpha + A_{12}\alpha^2 : A_{20} + A_{21}\alpha + A_{22}\alpha^2] \quad (4)$$

Here α is the parameter. The condition $\det(A) \neq 0$ guarantees not all coordinates vanish at once, so that Γ_A makes sense as a curve in $\boldsymbol{RP}^2$.

Lemma 2.1 *Γ_A is a nondegenerate conic section and the map $\alpha \to \Gamma_A(\alpha)$ is a rational diffeomorphism from $\boldsymbol{R} \cup \infty$ to this conic section.*

Proof: For any invertible matrix M, we have $M(\Gamma_A) = \Gamma_{MA}$. Choosing $M = A^{-1}$, we see that $M(\Gamma_A)$ is the conic section that intersects the affine patch in the parabola $x = y^2$. All the claims in the lemma are true for this special case, and then the general case follows from the fact that all the statements are invariant under projective transformations. ♠

2.2 Four Coincident Lines

In this section, as a warm-up, we study 4-tuples of coincident lines which otherwise satisfy the conditions of being nice. The most inportant thing in this section is the quantity D below. We cyclically relabel so that L_1 is not perpendicular to any other line. We translate so that all the lines contain the origin and rotate so that L_1 is the x-axis.

We first assume that L_2 and L_4 are not perpendicular. The 1-parameter group of dilations preserves L and so does reflection in the origin. The set of rectangles inscribed in L is invariant under these maps. If some rectangle inscribed in L has the origin as a vertex, we must have either (L_1, L_3) perpendicular or (L_2, L_4) perpendicular. So, for the case under consideration, the origin is not a vertex of an inscribed rectangle. Hence, every rectangle inscribed in L is a dilation/reflection of the ones having $(1, 0)$ as a vertex. We will show that there are 2 distinct inscribed rectangles having $(1, 0)$ as a vertex, and that the centers of these two rectangles lie on different lines through the origin. This implies that $C(L)$ is a union of 2 distinct lines (minus the origin).

For $j = 2, 3, 4$ the equation of L_j is given by $y_j = m_j x_j$. The values $0, m_2, m_3, m_4$ are pairwise distinct. Moreover, $1 + m_2 m_4 \neq 0$.

Define $T(x, y) = (-y, x)$. Note that (x, y) and $T(x, y)$ are perpendicular vectors. We seek a rectangle (R_1, R_2, R_3, R_4) inscribed in L having aspect ratio α. We set

$$R_1 = (1, 0), \qquad R_2 = (x_2, m_2 x_2),$$

$$R_3 = R_2 + \alpha T(R_2 - R_1), \qquad R_4 = R_1 + \alpha T(R_2 - R_1). \tag{5}$$

The conditions that $R_3 \in L_3$ and $R_4 \in L_4$ respectively lead to

$$\alpha = \frac{m_3 x_2 - m_2 x_2}{-1 + x_2(1 + m_2 m_3)}, \qquad \alpha = \frac{m_4}{-1 + x_2(1 + m_2 m_4)}, \tag{6}$$

Setting the two quantities equal and solving yields

$$x_2 = \frac{A \pm \sqrt{D}}{B}, \tag{7}$$

$$A = m_2 - m_3 - m_4 - m_2 m_3 m_4, \qquad B = 2(m_2 - m_3)(1 + m_2 m_4), \tag{8}$$

$$D = (m_2^2 + m_3^2 + m_4^2 + 2m_2 m_4 + 2m_2 m_3^2 m_4 + 4m_2^2 m_4^2 + m_2^2 m_3^2 m_4^2) - 2m_3(m_2 + m_4)(1 + m_2 m_4). \tag{9}$$

We call D the *positive discriminant*.

Lemma 2.2 $D > 0$.

Proof: Let $S \subset \boldsymbol{R}^3$ denote the set (m_2, m_3, m_4) where $m_2 \neq 0$ and $m_4 \neq 0$ and $m_2 m_4 + 1 \neq 0$. We let m_3 vary freely. The set S has 6 connected components, and the each of the 6 points

$$(\pm 1/2, 0, \pm 1/2), \qquad (\mp 1, \pm 1) \qquad (\pm 2, 0, \pm 2)$$

lies in a different component of S. We check explicitly that $D \in \{5/4, 4, 80\}$ at these points Solving $D = 0$ for m_3 yields

$$m_3 = \frac{(m_2 + m_4) \pm 2\ i\ m_2 m_4}{m_2 m_4 + 1}. \tag{10}$$

Hence there are no real solutions in S. This combines with our calculations at the 6 points to show that $D > 0$ everywhere. ♠

Returning to Equation 7, we note that $B \neq 0$ and $D > 0$. Hence there are always 2 distinct values of x_2 which yield an inscribed rectangle containing the point $(1, 0)$. Let R_1 and R_2 be the two rectangles. Suppose first that R_1 is centered at the origin. Then opposite vertices of R_1 lie on the same line through the origin, forcing our lines to coincide in pairs. This is a contradiction. The same argument works for R_2. Hence neither R_1 nor R_2 is centered at the origin. Suppose that the centers of R_1 and R_2 lie on the same line through the origin. Then there is some dilation λ such that R_1 and λR_2 have the same center, c. But then the order 2 rotation through c preserves all 4 lines and cycles the indices by 2 mod 4. This contradicts the fact that there are no parallel lines. This establishes our claims.

Now we consider the case when $m_2 m_4 + 1 = 0$. In this case the solution to Equation 6 is given by $x_2 = 1/(1 + m_2^2)$ and $a = 1/m_2$. A calculation shows that $R_3 = (0, 0)$. The corresponding inscribed rectangle has $(0, 0)$ as a vertex and is centered on L_1. Hence $L_1 - \{(0, 0)\} \subset C(L)$. By a similar construction, $L_3 - \{(0, 0)\} \subset C(L)$. This accounts for all the rectangles. Hence $C(L) = L_1 \cup L_3$ (minus the origin).

Remark: We proved that $D > 0$ under the assumption that $m_2 m_4 + 1 \neq 0$. When $m_2 m_4 + 1 = 0$ we have $D = 2 + m_2^2 + m_4^2 > 0$. So, the result holds in this special case as well.

2.3 The Regularity Test

Now we turn to the nice case, where the 4 lines of L do not contain a single point. The most important thing in this section is the quantity Δ below.

We can apply a dihedral symmetry to the labels so that L_1 is not perpendicular to any other line and L_3 does not contain $L_1 \cap L_2$. We rotate and scale the picture so that

- L_1 is the x-axis,
- L_2 contains $(0, 0)$,
- L_3 contains $(0, 1)$.

We have the following equations for the general point $(x_j, y_j) \in L_j$:

$$y_1 = 0, \qquad y_2 = m_2 x_2, \qquad y_3 = m_3 x_3 + 1, \qquad y_4 = m_4 x_4 + b_4. \tag{11}$$

In this section, we work out the condition that L is a regular configuration – i.e., the diagonals of L are nor perpendicular. We do this calculation by introducing vectors V_j so that $(x_j, y_j, 1) \cdot V_j = 0$ iff $(x_j, y_j) \in L_j$. These vectors are

$$V_1 = (0, 1, 0), \quad V_2 = (m_2, -1, 0), \quad V_3 = (m_3, -1, 1), \quad V_4 = (m_4, -1, b_4).$$

The point $L_i \cap L_j$ is given by projectivizing the cross product $V_i \times V_j$. When we work everything out, we find that

$$(L_{12} - L_{34}) \cdot (L_{23} - L_{41}) = \frac{\Delta}{(m_2 - m_3)(m_3 - m_4)m_4}, \tag{12}$$

$$\Delta = {b_4}^2(m_2 - m_3) + b_4(m_2 m_3 m_4 - m_2 + m_3 + m_4) + (-m_2 {m_4}^2 - m_4). \tag{13}$$

Since all the slopes are unequal, the configuration is regular if and only if $\Delta \neq 0$. We call Δ the *regularity test*.

Note that Δ is quadratic in b_4, so one can explicitly solve the equation $\Delta = 0$ in terms of b_4. A calculation shows that this quadratic equation has discriminant D, the positive discriminant we just finished studying. Hence, for each choice m_2, m_3, m_4, there are 2 distinct choice of b_4 which lead to singular configurations.

3 Proofs of the Results

3.1 A Matrix Equation

From now on, unless otherwise stated, L is a nice configuration. We normalize as in §2.3. We also recall that

$$T(x,y) = (-y,x). \tag{14}$$

This transformation has the property that V and $T(V)$ are perpendicular for any nonzero vector V.

We seek a rectangle (R_1,R_2,R_3,R_4) inscribed in L having aspect ratio α. We set

$$R_1 = (x_1,0), \qquad R_2 = (x_2, m_2x_2),$$

$$(x_3,y_3) = R_3 = R_2 + \alpha T(R_2 - R_1), \qquad (x_4,y_4) = R_4 = R_1 + \alpha T(R_2 - R_1).$$

The conditions that $(x_j,y_j) \in L_j$ for $j = 3,4$ lead to two equations in two unknowns:

$$M\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -b_4 \end{bmatrix}, \qquad M = \begin{bmatrix} \alpha & (m_3 - m_2) - \alpha(1 + m_2m_3) \\ \alpha + m_4 & -\alpha(1 + m_2m_4) \end{bmatrix} \tag{15}$$

This has a unique solution provided that $\det(M) \neq 0$. The equation $\det(M)$ is quadratic in α. The discriminant of this quadratic equation is D, the positive discriminant from §2.2. Hence there are always 2 values where $\det(M) = 0$.

In case α is a value where $\det(M) = 0$, the two rows of M are multiples of each other. We only get inscribed rectangles of aspect ratio α in this case if the second row of M is b_4 times the first row. This happens if and only if

$$\alpha = \frac{m_4}{b_4 - 1} = \frac{b_4m_3 - b_4m_2}{-1 + b_4 + m_2m_3 - m_2m_4}. \tag{16}$$

If $b_4 = 1$ there are no solutions. In general, solving this equation for b_4 leads to the same equation as solving $\Delta = 0$ for b_4. Here Δ is the regularity test. Hence there are additional rectangles inscribed in L corresponding to the values $\det(M) = 0$ if and only if L is singular.

We deal with the regular case in the next section and the singular case in the section following that.

3.2 Regular Configurations

In this section we prove the first half of Theorem 1.1. For α such that $\det(M) \neq 0$, the solution to Equation 15 is given by

$$x_1 = \frac{n_{10} + n_{11}\alpha}{d_0 + d_1\alpha + d_2\alpha^2}, \qquad x_2 = \frac{n_{20} + n_{21}\alpha}{d_0 + d_1\alpha + d_2\alpha^2}, \tag{17}$$

- $n_{10} = b_4(m_3 - m_2)$,
- $n_{11} = 1 - b_4 - b_4 m_2 m_3 + m_2 m_4$,
- $n_{20} = m_4$,
- $n_{21} = 1 - b_4$,
- $d_0 = m_4(m_2 - m_3)$,
- $d_1 = m_2 m_3 m_4 + m_2 - m_3 + m_4$,
- $d_2 = m_2(m_3 - m_4)$.

We remark that the discriminant of the equation $d_0 + d_1\alpha + d_2\alpha^2$ is, once again, the positive discriminant D from §2.2.

Recall that $C(L)$ is the curve of inscribed centers. In view of the work in the previous section, we get all points on $C(L)$ by analyzing Equation 17.

Lemma 3.1 *$C(L)$ is a nondegenerate conic section.*

Proof: Using the equations above for R_3 and R_4, we find that $C(L)$ is the curve Γ_A, where A is the matrix

$$\begin{matrix} b_4(m_3 - m_2) + m_4 & 2 - b_4(m_2 m_3 + 2) & (b_4 - 1)m_2 \\ m_2 m_4 & m_2 - b_4 m_3 + m_4 & m_2(b_4 m_3 - m_4) \\ 2(m_2 - m_3)m_4 & 2(m_3 m_4 m_2 + m_2 - m_3 + m_4) & 2m_2(m_3 - m_4) \end{matrix} \tag{18}$$

The condition $\det A \neq 0$ guarantees that $C(L)$ is a non-degenerate conic. We compute

$$\det(A) = 2m_2 m_3(m_2 - m_4)\Delta,$$

where Δ is the regularity test. Thus $\det(A) = 0$ if and only if L is a singular configuration. ♠

It remains to identity $C(L)$ as a hyperbola. We call $\alpha_0 \in \boldsymbol{R}$ an *asymptotic aspect ratio* if, for all $\epsilon > 0$, there is a rectangle R inscribed in L such that aspect ratio of R lies within ϵ of α_0 and the diameter of R exceeds $1/\epsilon$.

Lemma 3.2 *There exist* 2 *distinct asymptotic aspect ratios for* L*.*

Proof: We first recall that the discriminant the equation $d_0 + d_1\alpha + d_2\alpha_2 = 0$ is D, the positive discriminant. Hence this equation has 2 real roots. Let α_1 and α_2 be the two real roots of this equation. Since $\det(A) \neq 0$, it is impossible for α_1 to be a root of both equations $n_{j0} + n_{j1}\alpha = 0$. See §2.1. The same goes for α_2. Therefore, we can order our roots so that α_j is not a root of the equation $n_{j0} + n_{j1}\alpha_j = 0$. But than $x_1 \to \infty$ as $\alpha \to \alpha_1$. Since our lines have unequal slopes, the nearest point on $L_2 \cup L_3 \cup L_4$ from x_1 tends to ∞ as well. Hence, the diameter of the corresponding rectangle tends to ∞. At the same time, the aspect ratio converges to α_1. Hence α_1 is an asymptotic aspect ratio for L. The same argument works for α_2. ♠

Lemma 3.3 $L(C)$ *is neither an ellipse nor a parabola.*

Proof: Suppose $L(C)$ is an ellipse. Then we can find arbitrarily large rectangles inscribed in L and having center uniformly bounded. Rescaling the picture and taking a limit, we produce a new configuration L', consisting of 4 lines through the origin, having a nontrivial inscribed rectangle centered at the origin. The argument given in §2.2 rules this out.

Suppose that $L(C)$ is a parabola. Then we can find two rectangles $R_{n,1}$ and $R_{n,2}$ inscribed in L such that both rectangles have diameter n, and aspect ratios converging to the two asymptotic aspect ratios. Since the aspect ratios are uniformly bounded, the distance from the center of $R_{n,j}$ to the origin is of order n. Thus, we can take a rescaled limit so that both rescaled rectangles converge to finite rectangles R'_1 and R'_2 having distinct aspect ratio. Since the original rectangles have their centers on a parabola, R'_1 and R'_2 have their centers on the same line through the origin. Moreover, both R'_1 and R'_2 are inscribed in a configuration L' consisting of 4 lines through the origin. The argument given in §2.2 rules this out. ♠

Having eliminated the other possibilities, we see that $C(L)$ is a nondegenerate hyperbola. This completes the first half of Theorem 1.1.

3.3 Singular Configurations

In this section we prove the second half of Theorem 1.1.

Let M be the matrix from Equation 3.1. Even in the singular case, there are 2 values of α which make $\det(M) = 0$. However, in order for such a value to correspond to extra solutions not given by Equation 17, it must be the case that the second row of M is b_4 times the first row. This leads to the value

$$\alpha_0 = \frac{m_4}{b_4 - 1}.$$

Note that $b_4 \neq 1$ because in this case the line through $L_1 \cap L_2$ and $L_3 \cap L_4$ is vertical and the line through $L_2 \cap L_3$ and $L_1 \cap L_4$ is not. Hence α_0 is finite and nonzero. So, in the singular case, we really go get extra solutions.

Let is consider these extra solutions first. The locus of points (x_1, x_2) solving Equation 15 for α_0 is a straight line, and hence we can parametrize these solutions linearly. But then the vertices of R vary linearly and so does the center of R. Hence, the centers of the rectangles corresponding to these solutions exactly sweep out a line. Again, these rectangles all have aspect ratio α_0.

Comparing Equation 16 with Equation 17 we see that α_0 is a root of the numerator for the equation for x_1 and also a root of the numerator of the equation for x_2. If α_0 is not also the root of the denominators of these equations, then we have the solution $x_1 = x_2 = 0$, which gives an inscribed rectangle of diameter 0. This contradicts the fact that no 3 of our lines go through the same point. The only way out is that α_0 is also a root of the denominators of the Equations for x_1 and x_2.

Since the equations for x_1 and x_2 have a common root, we may factor it out. This gives linear equations for x_1 and x_2. Hence, the set of inscribed rectangles corresponding to solutions of Equation 17 have their centers on a straight line. Thus, we get a line of solutions having constant aspect ratio α_0 and a second line of solutions corresponding to solutions to Equation 15 and having varying aspect ratio.

Thus $C(L)$ is either two crossing lines or a repeated line. If $C(L)$ is a repeated line, then each point on this line is the center of 2 inscribed rectangles. This leads to the same contradiction as we have seen several times. Hence $C(L)$ consists of 2 crossing lines. This completes the proof of Theorem 1.1.

3.4 More Details about Regular Configurations

In this section we prove Theorem 1.2.

If we complete $C(L)$ to be a closed curve in $\boldsymbol{RP}^2$, then the parametrization which sends α to the center of the inscribed rectangle R_α of aspect ratio α is a rational diffeomorphism between $R \cup \infty$ and $C(L)$. See §2.1. The two asymptotic aspect ratios are mapped to the infinite points of $C(L)$. In particular, every point of the finite part of $C(L)$ is the center of a unique inscribed rectangle.

The discussion above shows that α is the aspect ratio of a rectangle inscribed in L if and only if α is not one of the two asymptotic aspect ratios. These two exceptional aspect ratios α_1 and α_2 are the two roots of the equation $d_0 + d_1\alpha + d_2\alpha^2$. Taking the product, we get

$$\alpha_1\alpha_2 = \frac{d_0}{d_2} = -\chi(L).$$

Here we are using the fact that the cross ratio is an affine invariant and we have rotated so that $m_1 = 0$. (To be sure, see Equations 20 and 22 below.)

It is worth noting that the solutions to the equation $d_0 + d_1\alpha + d_2\alpha^2$ only depend on the slopes of the lines of L and not on the Y-interecepts. Therefore, if we translate all the lines of L and produce a new configuration L', the two omitted aspect ratios do not change.

Now we discuss what happens for the slopes. We compute explicitly that there is a 2×2 matrix $\mu = \{\mu_{ij}\}$ such that

$$\sigma = \mu(\alpha) = \frac{\mu_{11}\alpha + \mu_{12}}{\mu_{21}\alpha + \mu_{22}}, \tag{19}$$

where

- $\mu_{11} = m_2 - m_2 b_4$,
- $\mu_{12} = m_2 m_4$,
- $\mu_{21} = m_2 m_3 b_4 - m_2 m_4$,
- $\mu_{22} = m_2 b_4 - m_3 b_4 + m_4$.

Solving the equation

$$\det \mu = \mu_{11}\mu_{22} - \mu_{12}\mu_{21} = 0$$

leads to the same equation as solving $\Delta = 0$. So, σ is the image of α under a nontrivial linear fractional transformation as long as L is a regular configuration. Because of this, the map $\sigma \to R_\sigma$ is again a rational diffeomorphism from $\boldsymbol{R} \cup \infty$ to the completed version of $C(L)$. Here R_σ is the rectangle having slope σ that is inscribed in L.

Finally, an explicit calculation, best done in a symbolic manipulator like Mathematica [**W**], shows that $\sigma_1\sigma_2 = -1$, when $\sigma_j = \mu(\alpha_j)$. This completes the proof of Theorem 1.2.

3.5 More Details about Singular Configurations

In this section we prove Theorem 1.4.

We have already seen that one line C_1 of $C(L)$ consists of centers of rectangles all having the same aspect ratio. If two such rectangles also have the same slope, then the homothety that takes the one rectangle to the other preserves the lines. This is only possible if the lines of L are parallel or all contain the same point. Under the assumptions of Theorem 1.4 this does not happen. Hence, the function $\sigma \to R_\sigma$ is an injective map from $\boldsymbol{R} \cup \infty$ to the completion of C_1 in the projective plane. The map is also continuous, given the equations we have for everything in sight. Hence, the map $\sigma \to \boldsymbol{R}_\sigma$ is a bijection. Therefore, every $\sigma \in \boldsymbol{R} \cup \infty$ except 1 arises as the slope of an inscribed rectangle centered on C_1.

Now we turn to the other line of $C(L)$.

Lemma 3.4 *The inscribed rectangles corresponding to solutions of Equation 17 are all parallel to each other.*

Proof: It is easier to give a geometric proof. What we want to show is that the rectangles of varying aspect ratio, whose centers all lie on the same line, are all parallel to each other. Given the algebraic nature of everything involved, it suffices to prove our claim for a small open set of 4-tuples of lines. We forget about our earlier normalization and now we choose L so that the lines of L extend the sides of a convex quadrilateral Q_L as shown in Figure 2. We insist that the diagonals of Q_L are horizontal and vertical.

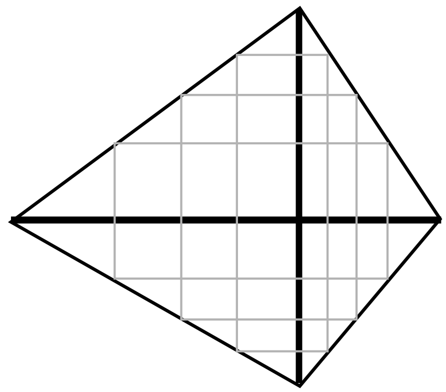

Figure 2: Rectangles inscribed in a singular configuration.

Now we play a kind of billiards game in Q, following along a horizontal line until we hit a side of Q and then going to a vertical side, then going to a horizontal side, and so on. It follows from similar triangles that all these trajectories are closed, and each one gives an inscribed rectangle. Hence, the inscribed rectangles in this case are all closed. ♠

Let C_2 be the other line of $C(L)$. The preceding result says that every point of C_2 is the center of a rectangle inscribed in L and all these rectangles are parallel to each other. To finish the proof of Theorem 1.4 we note that we can make the same kind of argument using aspect ratios with respect to C_2 that we made for slopes with respect to C_1. This completes the proof of Theorem 1.4.

3.6 Interpolating Between Rectangles

Each pair (R, R') of (labeled) rectangles determines a unique 4-tuple L of lines provided that each vertex of R is distinct from the corresponding vertex of R'. The line L_i contains the ith vertices of the 2 rectangles.

We call (R, R') *connected* if there is a continuous path of rectangles inscribed in L which connects R to R'. Corollary 1.3 says that (R, R') is disconnected provided that R and R' are perpendicular. Here we discuss a sharp result that is similar to Corollary 1.3.

Lemma 3.5 *Suppose that (R, R') determines a regular configuration of lines. then (R, R') is connected if and only if the pair $\{\alpha(R), \alpha(R')\}$ does not separate the pair $\{\alpha_{+1}, \alpha_{-1}\}$ on $\boldsymbol{R} \cup \infty$. Here $\alpha_{\pm 1}$ are as in Theorem 1.2.*

Proof: This result has the same proof as Corollary 1.3. ♠

In order to make Lemma 3.5 usable, we need a concrete way to compute α_{+1} and α_{-1}. These values are the roots of the equation $d_2\alpha^2 + d_1\alpha + d_0 = 0$ but above we only gave equations for these coefficients when $m_1 = 0$. This might not be so convenient in practice. Here we give the equations when m_1 is allowed to be arbitrary.

$$d_0 = +(m_2 - m_3)(m_4 - m_1). \tag{20}$$

$$d_1 = +(m_1 + m_3)(m_2m_4 - 1) - (m_2 + m_4)(m_1m_3 - 1). \tag{21}$$

$$d_2 = -(m_1 - m_2)(m_3 - m_4). \tag{22}$$

While we are at it, we give the more general form for the matrix μ from Equation 19. Now we make no constraint on the lines, except that we still require the configuration to be nice.

$$\mu_{11} = b_3m_1 - b_4m_1 - b_3m_2 + b_4m_2 - b_1m_3 + b_2m_3 + b_1m_4 - b_2m_4 \tag{23}$$

$$\begin{aligned}\mu_{12} = b_3m_1m_2 - b_4m_1m_2 - b_2m_1m_3 + b_4m_1m_3 +\\ b_1m_2m_4 - b_3m_2m_4 - b_1m_3m_4 + b_2m_3m_4\end{aligned} \tag{24}$$

$$\begin{aligned}\mu_{21} = -b_2m_1m_3 + b_4m_1m_3 + b_1m_2m_3 - b_4m_2m_3 +\\ b_2m_1m_4 - b_3m_1m_4 - b_1m_2m_4 + b_3m_2m_4\end{aligned} \tag{25}$$

$$\mu_{22} = -b_2m_1 + b_3m_1 + b_1m_2 - b_4m_2 - b_1m_3 + b_4m_3 + b_2m_4 - b_3m_4 \tag{26}$$

Generally speaking the equations are much more symmetric when we don't normalize the configuration. However, some of the expressions, such as the positive discriminant $D = d_1^2 - 4d_0d_2$, become too unwieldy to analyze easily. We normalized as above in order to get expressions of a manageable size.

4 Paths of Rectangles

4.1 The Convex Case

In this chapter we prove Theorem 1.6. We start with the convex case and then treat the nonconvex case.

Lemma 4.1 *Let L be a regular configuration. One can connect the diagonals of L by a continuous path of inscribed rectangles if and only if $\chi(L) < 0$.*

Proof: Two aspect ratios β_1 and β_2 correspond to rectangles lying on the different components of $C(L)$ if and only if the numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$ are interlaced. That is, the $(\alpha)s$ separate the $(\beta)s$ on $\boldsymbol{R} \cup \infty$. In particular, the diagonals of L, which correspond to $\beta_1 = 0$ and $\beta_2 = \infty$, are on the same component of $C(L)$ if and only if $\alpha_1\alpha_2 > 0$, which is true if and only if $\chi(L) < 0$. ♠

Let Q be a convex quadrilateral. The two diagonals of Q are maximal diameters. Q also has 2 saddles. These are diameters obtained by dropping a perpendicular from a vertex of Q to a side opposite that vertex. We want to see that a continuous path of rectangles inscribed in Q connects the maximal diameters to the saddles.

Let L_Q denote the configuration of 4 lines obtained by extending the sides of Q. All we need in the convex case is that L_Q is regular. We order the lines of Q according to the clockwise ordering of the edges of Q. We check for some particular example that $\chi(L_Q) > 0$. It follows from continuity that $\chi(L_Q) > 0$ for all examples.

By Lemma 4.1, there is no continous path of rectangles inscribed in L which connects the two diagonals of L. This fact alone does not prove Theorem 1.6. Recall that a rectangle R, by definition, is inscribed in L if each vertex of R is contained in a different edge of L. It might happen that some rectangle inscribed in Q has two vertices in the same side of Q. Such a rectangle is not, technically, inscribed in L. Figure 3 shows some examples of quadrilaterals like this.

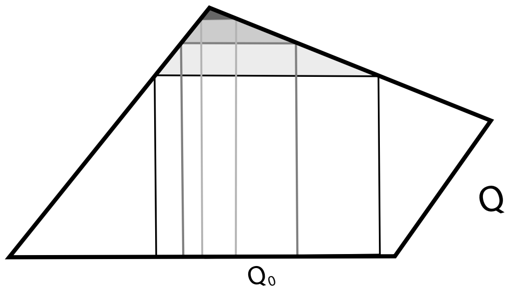

Figure 3: Some rectangles that are inscribed in Q but not L_Q.

So, suppose that R_t is a continuous path of rectangles inscribed in Q which supposedly connects the diagonals of Q. We can assume that R_t does not backtrack, so that the same rectangle does not appear twice. We set things up so that R_0 is one of the diagonals of Q and R_∞ is the other.

From what we have said above, there must be some $t > 0$ such that R_t is inscribed in Q but not in L_Q. Let's parametrize so that 1 is the supremal value such that R_t is inscribed in L for all $t \in [0, 1]$. This means that $R_{1+\epsilon}$ is not inscribed in L for all sufficiently small $\epsilon > 0$. Let $s = 1 + \epsilon$ and consider the picture for R_s.

One edge E_s of R_s lies in a single edge Q_0 of Q and the opposite edge E'_s cuts off a triangle of Q, just as in Figure 3. As s increases, E_s is stuck on Q_0 (because there is no backtracking) and the only option is for E_s to shrink to a point on Q_0. At the same time, the opposite edge E'_s shrinks to a vertex of Q. The chord joining these two limiting points is a saddle diameter. Our path never reaches the other diagonal of Q. This completes the proof of Theorem 1.6 in the convex case.

4.2 The Nonconvex Case

Let Q be nonconvex and as in the hypotheses of Theorem 1.6. This time one of the diagonals of Q is maximal and the other one is a saddle. See Figure 4. We deal with these diameters first.

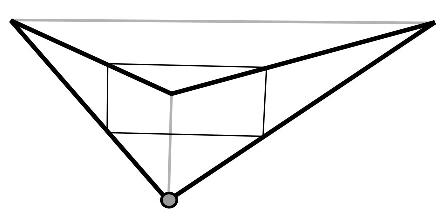

Figure 4: The diagonals of Q and an inscribed rectangle.

Lemma 4.2 *There is a path of rectangles inscribed in Q which connects the two diagonals.*

Proof: Let L_Q be the 4-tuple of lines obtained by extending the sides of Q, as above. This time we compute that $\chi(L_Q) < 0$. So, there is a continuous path of rectangles inscribed in L_Q that connects the two diagonals of Q. Figure 4 shows one of the rectangles along this path. Very near the diagonals, the rectangles in the path are inscribed in Q. If some rectangle on our path is not inscribed in Q, then by continuity, some nondegenerate rectangle R has the following properties.

- R is inscribed in Q.
- R is inscribed in L_Q.
- A vertex of R is also a vertex of Q.

A case-by-case analysis shows that there is no such rectangle like this. For instance, if v is the bottom vertex in FIgure 4, then one side of R must have length 0. Given that the properties just listed are impossible, our path of rectangles remains inscribed in Q and connects the two diagonals of Q. ♠

One of the interior angles of Q is obtuse. We call the opposite vertex the *dart vertex*. The bottom vertex in Figure 4 is the dart vertex. We call the interior angle at the dart vertex the *dart angle*. If the dart angle is obtuse, there are no other diameters of Q. In this case Theorem 1.6 is automatically true. We don't even need Lemma 4.2.

If the dart angle is acute, there are 4 more diameters of Q. The two additional maximal diameters are the sides of Q adjacent to the dart vertex. The two additional saddles are obtained by dropping perpendiculars from each of the dart-adjacent vertices to the opposite sides. Figure 5 shows an example of one of the maximal diameters D and one of the saddles S.

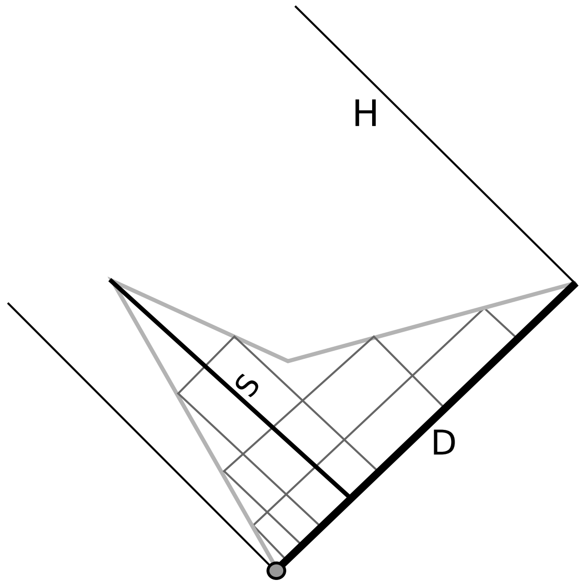

Figure 5: Diameters D, S and a rectangle path connecting them.

Because the convex hull of Q is a triangle, and triangles cannot have two obtuse angles, at least one of the two maximal diameters D has the property that all of Q is contained in the half-strip H whose base is D and whose sides are perpendicular to D. This is shown in Figure 5. In this case, the path of rectangles starting from D always has a side contained in D, and shrinks down to S. So, there is a path of rectangles inscribed in Q connecting D to S.

The argument above might break down for the other pair of diameters. However, we have only one pair left and they must be connected to each other by a process of elimination. Let us elaborate on this point. Looking sufficiently near each diameter, we see that there is only one path of rectangles which limits to that diameter. So, by process of elimination, any remaining path of rectangles connecting two diameters must connect the last two diagonals. Again, one of these diameters is a maximum and one is a saddle.

This completes the proof of Theorem 1.6.

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