A simple proof of the cross-ratio inequality.

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We give a simple proof of the main inequality from [1].

1 The inequality

For any positive integer $n \in \mathbb{N}$ the following inequality is true:

$$\prod_{i=1}^{n}\Big(1+a_i+a_{i+1}\Big) \geq \prod_{i=1}^{n}\Big((1+a_{i-1})\cdot(1+a_{i+1})-a_i^2\Big) \qquad \forall (a_1, a_2, \dots, a_n) \in [0,1]^n.$$

where the indices are taken $\mod n$. Moreover, equality holds if and only if all a_i are equal.

Remark. We denote by LHS and RHS, the left hand side and right hand side in the inequality above, and use this notation in the rest of this note.

1.1 A simpler inequality

Let $(x, y, z) \in [0,1]^3$. Using the Cauchy-Schwartz inequality we obtain the following.

$1 \leq \sqrt{(1+x)(1+y)} \leq \dfrac{1+x+1+y}{2}$ and $z \leq 1$ imply that:

(A) $(\sqrt{(1+x)(1+y)} - z)^2 \leq (1 + \dfrac{x+y}{2} - z)^2$.

Similarly we obtain:

(B) $(\sqrt{(1+x)(1+y)} + z)^2 \leq (1+x+z)\cdot(1+y+z)$.

Multiplying (A) and (B) proves :

$$\textbf{(C)} \qquad \frac{\big((1+x)\cdot(1+y)-z^2\big)^2}{(1+x+z)\cdot(1+y+z)} \leq \big(1+\frac{x+y}{2}-z\big)^2$$

2 The proof

We apply inequality (C) n times, one for each triple $(x,y,z)=(a_{i-1},a_{i+1},a_i)$, and multiply all of them together to obtain :

$$\textbf{(D)}\qquad \big(\frac{\text{RHS}}{\text{LHS}}\big)^2 \leq \Big(\prod_{i=1}^{n}(1+\frac{a_{i-1}+a_{i+1}}{2}-a_i)\Big)^2.$$

Finally, we note that $\frac{a_{i-1}+a_{i+1}}{2}-a_i \geq -1$ and that the function $f(x)=\log(1+x)$ is concave for $x\geq -1$. Therefore we can apply Jensen's inequality to obtain :

$$\textbf{(E)}\qquad \prod_{i=1}^{n}(1+\frac{a_{i-1}+a_{i+1}}{2}-a_i) \leq \big(1+\frac{\sum_{i=1}^{n}\frac{a_{i-1}+a_{i+1}}{2}-a_i}{n}\big)^n = 1.$$

Putting (D) and (E) together completes the proof.

References

[1] R. E. Schwartz *A Conformal Averaging Process on the Circle.*