An Improved Bound on the Optimal Paper Mobius Band

Richard Evan Schwartz *

July 19, 2020

Abstract

We show that a smooth embedded paper Mobius band must have aspect ratio at least

$$\lambda_1 = \frac{2\sqrt{4 - 2\sqrt{3}} + 4}{\sqrt{3}\sqrt{2} + 2\sqrt{2\sqrt{3} - 3}} = 1.69497...$$

This bound comes more than 3/4 of the way from the old known bound of $\pi/2 = 1.5708...$ to the conjectured bound of $\sqrt{3} = 1.732...$.

1 Introduction

In §14 of their book, Mathematical Omnibus [FT], Dmitry Fuchs and Sergei Tabachnikov give a beautiful exposition of what is known about the following question: What is the aspect ratio of the shortest smooth paper Mobius band? Let’s state this more precisely. Given $\lambda > 0$, let

$$M_\lambda = ([0,1] \times [0,\lambda]) / \sim, \quad (x,0) \sim (1-x, \lambda)$$

(1)

denote the standard flat Mobius band of width 1 and height $\lambda$. This Mobius band has aspect ratio $\lambda$. Let $S \subset \mathbb{R}_+$ denote the set of values of $\lambda$ such that there is a smooth isometric embedding $I : M_\lambda \to \mathbb{R}^3$. The question above asks for the quantity

$$\lambda_0 = \inf S.$$

(2)

*Supported by N.S.F. Grant DMS-1807320

1
In [FT] it is pointed out that the smooth requirement (or some suitable variant) is necessary in order to have a nontrivial problem. Given any $\epsilon > 0$, one can start with the strip $[0, 1] \times [0, \epsilon]$ and first fold it (across vertical folds) so that it becomes, say, an $(\epsilon/100) \times \epsilon$ “accordion”. One can then easily twist this “accordion” once around in space so that it makes a Mobius band. The corresponding map from $M_\epsilon$ is an isometry but not smooth.

In [FT] it is proved that $\lambda_0 \in [\pi/2, \sqrt{3}]$. The lower bound is local in nature and does not see the difference between immersions and embeddings. Indeed, in [FT], a sequence of immersed examples whose aspect ratio tends to $\pi/2$ is given. The upper bound comes from an explicit construction. The left side of Figure 1.1 shows $M_{\sqrt{3}}$, together with a certain union of bends drawn on it. The right side shows the nearly embedded paper Mobius band one gets by folding this paper model up according to the bending lines. We recommend that the reader try to make one.

![Figure 1.1: The conjectured optimal paper Mobius band](image)

The Mobius band just described is degenerate: It coincides as a set with the equilateral triangle $\Delta$ of sidelengh 2/$\sqrt{3}$. However, one can choose any $\epsilon > 0$ and find a nearby smoothly embedded image of $M_{\sqrt{3} + \epsilon}$ by a process of rounding out the folds and slightly separating the sheets. The conjecture is that $\lambda_0 = \sqrt{3}$, so that the triangular example is the best one can do.

In this paper we improve the lower bound but don’t quite get all the way to the conjectured lower bound.

**Theorem 1.1 (Main)** An embedded paper Mobius band must have aspect ratio at least

$$\lambda_1 = \frac{2\sqrt{4 - 2\sqrt{3}} + 4}{\sqrt{3}\sqrt{2} + 2\sqrt{2\sqrt{3} - 3}} = 1.69497...$$

This value $\lambda_1$ arises naturally in a geometric optimization problem involving trapezoids.
The proof of the Main Theorem has 2 ideas, which we now explain. Being a ruled surface, \( I(M_\lambda) \) contains a continuous family of line segments which have their endpoints on \( \partial I(M_\lambda) \). We call these line segments \emph{bend images}. Say that a \( T \)-pattern is a pair of perpendicular coplanar bend images. The \( T \)-pattern looks somewhat like the two vertical and horizontal segments on the right side of Figure 1.1 except that the two segments are disjoint in an embedded example. Here is our first idea.

\textbf{Lemma 1.2} An embedded paper Mobius band of aspect ratio less than \( 7\pi/12 \) contains a \( T \)-pattern.

Note that \( 7\pi/12 > \sqrt{3} \), so Lemma 1.2 applies to the examples of interest to us. Lemma 1.2 relies crucially on the embedding property. The immersed examples in [FT] do not have these \( T \)-patterns.

The two bend images comprising the \( T \)-pattern divide \( I(M) \) into two halves. Our second idea is to observe that the image \( I(\partial M_\lambda) \) makes a loop which hits all the vertices of the \( T \)-pattern. When we compare the relevant portions of \( \partial M_\lambda \) with the convex hull of the vertices of the \( T \)-pattern, we get two constraints which lead naturally to the lower bound of \( \lambda_1 \).

It would have been nice if the existence of a \( T \)-pattern forced \( \lambda > \sqrt{3} \) on its own. This is not the case. In Figure 2.1 we show half of an immersed paper Mobius band of aspect ratio less than \( \sqrt{3} \) that has a \( T \)-pattern. I did not try to formally prove that these exist, but using my computer program I can construct them easily. The lowest value of \( \lambda \) I can get is around \( \sqrt{3} - .012 \).

In §2 we introduce some basic geometric objects associated to a paper Mobius band and the prove Lemma 1.2. In §3 we give the convex hull argument mentioned above. At the end of §3 we prove an auxiliary result using similar methods.

I would like to thank Sergei Tabachnikov for telling me about this problem and for helpful discussions about it.
2 Existence of the T Pattern

2.1 Polygonal Mobius Bands

**Basic Definition:** Say that a *polygonal Mobius band* is a pair \( \mathcal{M} = (\lambda, I) \) where \( I : M_\lambda \to \mathbb{R}^3 \) is an isometric embedding that is affine on each triangle of a triangulation of \( M_\lambda \). We insist that the vertices of these triangles all lie on \( \partial M_\lambda \), as in Figure 1.1. Any smooth isometric embedding \( I' : M_\lambda \to \mathbb{R}^3 \) can be approximated arbitrarily closely by this kind of map, so it suffices to work entirely with polygonal Mobius bands.

**Associated Objects:** Let \( \delta_1, ..., \delta_n \) be the successive triangles of \( \mathcal{M} \).

- The *ridge* of \( \delta_i \) is edge of \( \delta_i \) that is contained in \( \partial M_\lambda \).
- The *apex* of \( \delta_i \) to be the vertex of \( \delta_i \) opposite the ridge.
- A *bend* is a line segment of \( \delta_i \) connecting the apex to a ridge point.
- A *bend image* is the image of a bend under \( I \).
- A *facet* is the image of some \( \delta_i \) under \( I \).

We always represent \( M_\lambda \) as a parallelogram with top and bottom sides identified. We do this by cutting \( M_\lambda \) open at a bend. See Figure 2.1 below.

**The Sign Sequence:** Let \( \delta_1, ..., \delta_n \) be the triangles of the triangulation associated to \( \mathcal{M} \), going from bottom to top in \( P_\lambda \). We define \( \mu_i = -1 \) if \( \delta_i \) has its ridge on the left edge of \( P_\lambda \) and \( +1 \) if the ridge is on the right. The sequence for the example in Figure 1.1 is \(+1, -1, +1, -1\).

**The Core Curve:** There is a circle \( \gamma \) in \( M_\lambda \) which stays parallel to the boundary and exactly \( 1/2 \) units away. In Equation 1, this circle is the image of \( \{1/2\} \times [0, \lambda] \) under the quotient map. We call \( I(\gamma) \) the *core curve*.

The left side of Figure 2.1 shows \( M_\lambda \) and the pattern of bends. The vertical white segment is the bottom half of \( \gamma \). The right side of Figure 2.1 (which has been magnified to show it better) shows \( I(\tau) \) where \( \tau \) is the colored half of \( M_\lambda \). All bend angles are \( \pi \) and the whole picture is planar. The colored curve on the right is the corresponding half of the core curve. Incidentally, for \( \tau \) we have \( L + R = 1.72121... < \sqrt{3} \).
The Ridge Curve: We show the picture first, then explain.

Figure 2.1: The bend pattern and the bottom half of the image

Figure 2.2: Half 2x core curve (red/blue) and half ridge curve (black).
Let $\beta_b$ be the bottom edge of the parallelogram representing $M_\lambda$. We normalize so that $I$ maps the left vertex of $\beta_b$ to $(0, 0, 0)$ and the right vertex to $(B, 0, 0)$, where $B$ is the length of $\beta_b$. Let $E_1, \ldots, E_n$ be the successive edges of the core curve, treated as vectors. Let
\[ \Gamma'_i = 2\mu_i E_i, \quad i = 1, \ldots, n. \]

Let $\Gamma$ be the curve whose initial vertex is $(B, 0, 0)$ and whose edges are $\Gamma'_1, \ldots, \Gamma'_n$. Here $\mu_1, \ldots, \mu_n$ is the sign sequence.

$\Gamma$ has length $2\lambda$, connects $(B, 0, 0)$ to $(-B, 0, 0)$, and is disjoint from the open unit ball. The lines extending the sides of $\Gamma$ are tangent to the unit sphere. We rotate so that $\Gamma$ contains $(0, T, 0)$ for some $T > 1$. If we cone $\Gamma$ to the origin, we get a collection $\Delta_1, \ldots, \Delta_n$ of triangles, and $\Delta_i$ is the translate of $\mu_i I(\delta_i)$ whose apex is at the origin. In particular, the vectors pointing to the vertices of $\Gamma$ are parallel to the corresponding bend images. Figure 2.2 shows the portion of the ridge curve (in black) associated to the example in Figure 2.1. We have also scaled the core curve by 2 and translated it to show the relationships between the two curves.

### 2.2 Geometric Bounds

While we are in the neighborhood, we re-prove the lower bound from [FT]. The proof in [FT] is somewhat similar, though it does not use the ridge curve. Let $\lambda$ be the aspect of the polygonal Mobius band $M$ and let $\Gamma$ be the associated ridge curve. Let $f : \mathbb{R}^3 - B^3 \to S^2$ be orthogonal projection. The map $f$ is arc-length decreasing. Letting $\Gamma^* = f(\Gamma)$, we have $|\Gamma^*| < |\Gamma| = 2\lambda$. Since $\Gamma^*$ connects a point on $S^2$ to its antipode, $|\Gamma^*| \geq \pi$. Hence $\lambda > \pi/2$.

Now we use the same idea in a different way.

**Lemma 2.1** Suppose $M$ has aspect ratio less than $7\pi/12$. Then the ridge curve $\Gamma$ lies in the open slab bounded by the planes $Z = \pm 1/\sqrt{2}$.

**Proof:** We divide $\Gamma$ into halves. One half goes from $(B, 0, 0)$ to $(0, T, 0)$ and the second half goes from $(0, T, 0)$ to $(-B, 0, 0)$. Call the first half $\Gamma_1$. Suppose that $\Gamma_1$ intersects the plane $Z = 1/\sqrt{2}$. Then the spherical projection $\Gamma'_1$ goes from $A = (1, 0, 0)$ to some unit vector $B = (u, v, 1/\sqrt{2})$ to $C = (0, 1, 0)$. Here $u^2 + v^2 = 1/2$. The shortest path like this is the geodesic bigon connecting $A$ to $B$ to $C$. Such a bigon has length at least
\[ \arccos(A \cdot B) + \arccos(B \cdot C) = \arccos(u) + \arccos(v) \geq 2 \arccos(1/2) = 2\pi/3. \]
The starred inequality comes from the fact that the minimum, subject to the constraint \( u^2 + v^2 = 1/2 \), occurs at \( u = v = 1/2 \).

But then \( \Gamma \) has length at least \( 2\pi/3 + \pi/2 = 7\pi/6 \). This exceeds twice the aspect ratio of \( M \). This is a contradiction. The same argument works if \( \Gamma_1 \) hits the plane \( Z = -1/\sqrt{2} \). Likewise the same argument works with the second half of \( \Gamma \) in place of the first half. ♠

**Corollary 2.2** Suppose \( M \) has aspect ratio less than \( 7\pi/12 \). Let \( \beta_1^* \) and \( \beta_2^* \) be two perpendicular bend images. Then a plane parallel to both \( \beta_1^* \) and \( \beta_2^* \) cannot contain a vertical line.

**Proof:** Every bend image is parallel to some vector from the origin to a point of \( \Gamma \). By the previous result, such a vector make an angle of less than \( \pi/4 \) with the \( XY \)-plane. Hence, all bend images make angles of less than \( \pi/4 \) with the \( XY \)-plane. Suppose our claim is false. Since \( \beta_1^* \) and \( \beta_2^* \) are perpendicular to each other, one of them must make an angle of at least \( \pi/4 \) with the \( XY \)-plane. This is a contradiction. ♠

### 2.3 Perpendicular Lines

As a prelude to the work in the next section, we prove a few results about lines and planes. Say that an anchored line in \( \mathbb{R}^3 \) is a line through the origin. Let \( \Pi_1 \) and \( \Pi_2 \) be planes through the origin in \( \mathbb{R}^3 \).

**Lemma 2.3** Suppose that \( \Pi_1 \) and \( \Pi_2 \) are not perpendicular. The set of perpendicular anchored lines \( (L_1, L_2) \) with \( L_j \in \Pi_j \) for \( j = 1, 2 \) is diffeomorphic to a circle.

**Proof:** For each anchored line \( L_1 \in \Pi_1 \) the line \( L_2 = L_1^\perp \cap \Pi_2 \) is the unique choice anchored line in \( \Pi_2 \) which is perpendicular to \( L_1 \). The line \( L_2 \) is a smooth function of \( L_1 \). So, the map \( (L_1, L_2) \rightarrow L_1 \) gives a diffeomorphism between the space of interest to us and a circle. ♠

A sector of the plane \( \Pi_j \) is a set linearly equivalent to the union of the \((++)\) and \((-\)) quadrants in \( \mathbb{R}^2 \). Let \( \Sigma_j \subset \Pi_j \) be a sector. The boundary \( \partial \Sigma_j \) is a union of two anchored lines.
Lemma 2.4 Suppose (again) that the planes $\Pi_1$ and $\Pi_2$ are not perpendicular. Suppose also that no line of $\partial \Sigma_1$ is perpendicular to a line of $\partial \Sigma_2$. Then the set of perpendicular pairs of anchored lines $(L_1, L_2)$ with $L_j \in \Sigma_j$ for $j = 1, 2$ is either empty or diffeomorphic to a closed line segment.

Proof: Let $S^1$ denote the set of perpendicular pairs as in Lemma 2.3. Let $X \subset S^1$ denote the set of those pairs with $L_j \in \Sigma_j$. Let $\pi_1$ and $\pi_2$ be the two diffeomorphisms from Lemma 2.3. The set of anchored lines in $\Sigma_j$ is a line segment and hence so is its inverse image $X_j \subset S^1$ under $\pi_j$. We have $X = X_1 \cap X_2$. Suppose $X$ is nonempty. Then some $p \in X$ corresponds to a pair of lines $(L_1, L_2)$ with at most one $L_j \in \partial \Sigma_j$. But then we can perturb $p$ slightly, in at least one direction, so that the corresponding pair of lines remains in $\Sigma_1 \times \Sigma_2$. This shows that $X_1 \cap X_2$, if nonempty, contains more than one point. But then the only possibility, given that both $X_1$ and $X_2$ are segments, is that their intersection is also a segment. ♠

2.4 The Space of Perpendicular Pairs

We prove the results in this section more generally for piecewise affine maps $I : M_\lambda \to \mathbb{R}^3$ which are not necessarily local isometries. The reason for the added generality is that it is easier to make perturbations within this category. Let $X$ be the space of such maps which also satisfy the conclusion of Corollary 2.2. (In this section we will not use this property but in the next section we will.) So, $X$ includes all (isometric) polygonal Mobius bands of aspect ratio less than $7\pi/12$. The notions of bend images and facets makes sense for members of $X$.

Lemma 2.5 The space $X$ has a dense set $Y$ which consists of members such that no two facets lie in perpendicular planes and no two special bend images are perpendicular.

Proof: One can start with any member of $X$ and postcompose the whole map with a linear transformation arbitrarily close to the identity so as to get a member of $Y$. The point is that we just need to destroy finitely many perpendicularity relations. ♠
Let $\gamma$ be center circle of $M_\lambda$. We can identify the space of bend images of $\mathcal{M}$ with $\gamma$: The bends and bend-images are in bijection, and each bend intersects $\gamma$ once. The space of ordered pairs of unequal bend images can be identified with $\gamma \times \gamma$ minus the diagonal. We compactify this space by adding in 2 boundary components. One of the boundary components comes from approaching the main diagonal from one side and the other comes from approaching the diagonal from the other side. The resulting space $A$ is an annulus. For the rest of the section we choose a member of $Y$ and make all definitions for this member.

**Lemma 2.6** $\mathcal{P}$ is a piecewise smooth 1-manifold in $A$.

**Proof:** We apply Lemma 2.4 to the planes through the origin parallel to the facets and to the anchored lines parallel to the bend images within the facets. (Within a single facet the bend images and the corresponding anchored lines are in smooth bijection.) By Lemma 2.4, the space $\mathcal{P}$ is the union of finitely many smooth connected arcs. Each arc corresponds to an ordered pair of facets which contains at least one point of $\mathcal{P}$. Each of these arcs has two endpoints. Each endpoint has the form $(\beta_1^*, \beta_2^*)$ where exactly one of these bend images is special. Let us say that $\beta_1^*$ is special. Then $\beta_1^*$ is the edge between two consecutive facets, and hence $(\beta_1^*, \beta_2^*)$ is the endpoint of exactly 2 of the arcs. Hence the arcs fit together to make a piecewise smooth 1-manifold. ♠

A component of $\mathcal{P}$ is **essential** if it separates the boundary components of $A$.

**Lemma 2.7** $\mathcal{P}$ has an odd number of essential components.

**Proof:** An essential component, being embedded, must represent a generator for the first homology $H_1(A) = \mathbb{Z}$. By duality, a transverse arc running from one boundary component of $A$ to the other intersects an essential component an odd number of times and an inessential component an even number of times. Let $a$ be such an arc. As we move along $a$ the angle between the corresponding bends can be chosen continuously so that it starts at 0 and ends at $\pi$. Therefore, $a$ intersects $\mathcal{P}$ an odd number of times. But this means that there must be an odd number of essential components of $\mathcal{P}$. ♠
2.5 The Main Argument

Now we prove Lemma 1.2. Say that a member of $X$ is *good* if (with respect to this member) there is a path connected subset $K \subset P$ such that both $(\beta_1^*, \beta_2^*)$ and $(\beta_2^*, \beta_1^*)$ belong to $K$ for some pair $(\beta_1^*, \beta_2^*)$.

**Lemma 2.8** If $M$ is good then $M$ has a T-pattern.

**Proof:** Each pair $(\beta_1^*, \beta_2^*)$ in $P$ determines a unique pair of parallel planes $(P_1, P_2)$ such that $\beta_j^* \subset P_j$ for $j = 1, 2$. By hypothesis, members of these planes do not contain vertical lines. Hence, our planes intersect the $Z$-axis in single and continuously varying points. As we move along $K$ these planes exchange places and so do their $Z$-intercepts. So, at some instant, the planes coincide and give us a T-pattern. ♠

**Lemma 2.9** A dense set of members of $X$ are good.

**Proof:** Let $Y$ be the dense subset of $X$ considered in the previous section. Relative to any member of $Y$, the space $P$ is a piecewise smooth 1-manifold of the annulus $A$ with an odd number of essential components. The involution $\iota$, given by $\iota(p_1, p_2) = (p_2, p_1)$, is a continuous involution of $A$ which preserves $P$ and permutes the essential components. Since there are an odd number of these, $\iota$ preserves some essential component of $P$. But then this essential component contains our set $K$. ♠

Now we know that there are T-patterns for members of a dense subset of $X$. By compactness and continuity, every member of $X$ has a T-pattern. By Corollary 2.2, $X$ contains all (ordinary) embedded polygonal Mobius bands of aspect ratio less than $7\pi/12$. Hence all such embedded Mobius bands contain T-patterns. This completes the proof of Lemma 1.2.
3 The Length Bound

3.1 Constraints coming from the T Pattern

Let $\mathcal{M}$ be a polygonal Mobius band of aspect ratio $\lambda < 7\pi/12$. We keep the notation from the previous chapter.

Let $\beta_1$ and $\beta_2$ be two bends whose corresponding images $\beta_1^* = I(\beta_1)$ and $\beta_2^* = I(\beta_2)$ form a $T$-pattern. Since these segments do not intersect, we can label so that the line extending $\beta_2^*$ does not intersect $\beta_1^*$. We cut $M_\lambda$ open along $\beta_1$ and treat $\beta_1$ as the bottom edge. We now set $\beta_b = \beta_1$ and $\beta_t = \beta_2$ and (re)normalize as in §2.1. So, $\beta_b^*$ connects $(0,0,0)$ to $(B,0,0)$, and $\beta_t^*$ is a translate of the segment connecting $(0,0,0)$ to $(0,T,0)$. Here $B$ and $T$ are the lengths of these segments.

The left side of Figure 3.1 shows $M_\lambda$. Reflecting in a vertical line, we normalize so that $L_1 \geq R_1$. This means that $L_2 \geq R_2$. The right side of Figure 3.1 shows that $T$ pattern, and the corresponding images of the sets on the left under the isometry $I$. The wiggly curves we have drawn do not necessarily lie in the $XY$-plane but their endpoints do.

![Figure 3.1: A Paper Mobius band interacting with the T-pattern.](image)

There is some $\epsilon$ such that the distance from the white vertex to the yellow vertex is $T/2 + \epsilon$ and the distance from the white vertex to the yellow vertex is $T/2 - \epsilon$. Looking at the picture, and using the fact that geodesics in the Euclidean plane are straight lines, we get the following constraints:

\[ R_1 + R_1 \geq T, \]  \hspace{1cm} (4)

\[ L_1 + L_2 \geq \sqrt{B^2 + (T/2 - \epsilon)^2} + \sqrt{B^2 + (T/2 + \epsilon)^2} \geq 2\sqrt{B^2 + T^2/4}. \]  \hspace{1cm} (5)
3.2 An Optimization Problem

We are done with the Mobius band. We just have a parallelogram as on
the left side of Figure 3.1 which satisfies the constraints in Equations 4 and
5 and we want to minimize $L_1 + L_2 + R_1 + R_2$. Let $L = (L_1 + L_2)/2$
and $R = (R_1 + R_2)/2$. If we replace $L_1, L_2$ by $L, L$ and $R_1, R_2$ by $R, R$
the constaints are still satisfied and the sum of interest is unchanged. The
constraints now become:

$$2R \geq T, \quad L \geq \sqrt{B^2 + T^2/4}.$$  \hspace{1cm} (6)

We show that $L + R \geq \lambda_1$, the constant from the Main Theorem, which
means

$$\lambda = \frac{1}{2}(L_1 + L_2 + R_1 + R_2) = L + R \geq \lambda_1.$$  \hspace{1cm}

So, showing that $L + R \geq \lambda_1$ finishes the proof of the Main Theorem.

Let $b$ and $t$ respectively denote the slopes of the sides labeled $B$ and $T$
in Figure 3.2. Let $S = L + R$. Since $L \geq R$ we have $b \geq t$. In Figure 3.2
we depict the case when $b > 0$ and $t < 0$. As we see in the next section, this
must happen when $S < \sqrt{3}$.

![Figure 3.2: The basic trapezoid.](image)

Since $L + R = S$ and $L - R = b - t$, and by the Pythagorean Theorem,

$$L = \frac{S + b - t}{2}, \quad R = \frac{S - b + t}{2}, \quad B = \sqrt{1 + b^2}, \quad T = \sqrt{1 + t^2}.$$

(7)

Plugging these relations into Equation 6, we get $S \geq f(b, t)$ and $S \geq g(b, t)$
where

$$f(b, t) = b - t + T, \quad g(b, t) = -b + t + \sqrt{4B^2 + T^2}.$$  \hspace{1cm}

Let $\phi = \max(f, g)$ and let $D$ be the domain where $b \geq t$. We have
$S \geq \phi$. To finish the proof of the Main Theorem we just need to show that
$\min_D \phi = \lambda_1$. This is what we do.
Lemma 3.1 \( \phi \) achieves its minimum at a point in the interior of \( D \), and at this point we have \( f = g \).

Proof: We note 3 properties of our functions:
1. \( f(0, -1/\sqrt{3}) = g(0, -1/\sqrt{3}) = \sqrt{3} \). Hence \( \phi(0, -1/\sqrt{3}) = \sqrt{3} \).
2. On \( \partial D \) we have \( g(b, b) = B\sqrt{5} \geq \sqrt{5} \). Hence \( \min_{\partial D} \phi > \sqrt{3} \).
3. As \( b^2 + t^2 \to \infty \) in \( D \), we have \( f, g \to +\infty \).

From these properties, we see that the global minimum of \( \phi \) is achieved at some point in the interior of \( D \). Next, we compute
\[
\frac{\partial f}{\partial t} = -1 + \frac{t}{T} < 0, \quad \frac{\partial g}{\partial t} = 1 + \frac{t}{4B^2 + T^2} > 0.
\]
Hence the gradients of \( f \) and \( g \) are never zero, and are never positive multiples of each other. For this reason, any point where \( \phi \) achieves a global minimum lies on the set where \( f = g \). ♠

Setting \( f = g \) and solving for \( b \), we get
\[
b = \beta(t) = \frac{t^3 - T^3 - 3t}{3t^2 - 1}. \tag{8}
\]
There no solutions to \( f = g \) when \( t = +1\sqrt{3} \) and Item 1 above (or a direct calculation) shows that \( \beta(-1/\sqrt{3}) = 0 \). In particular, \( \beta \) has a removable singularity at \( t = -1/\sqrt{3} \) and hence is smooth on the domain
\[
D^* = (-\infty, 1/\sqrt{3}).
\]
When \( t > 1/\sqrt{3} \) we have \( b = \beta(t) < 0 < t \). Hence, only \( t \in D^* \) corresponds to points in \( D \). So, we just need show \( \min_{D^*} \phi^* = \lambda_1 \), where
\[
\phi^*(t) = f(\beta(t), t) = \frac{2T(t^2 - tT - 1)}{3t^2 - 1}. \tag{9}
\]
As \( t \to -\infty \) or \( t \to 1/\sqrt{3} \) we have \( \phi^*(t) \to +\infty \). Hence \( \phi^* \) achieves its minimum on \( D^* \) at some point where \( d\phi^*/dt = 0 \). This happens only at
\[
t_0 = -\sqrt{\frac{2}{\sqrt{3}}} - 1 = -0.39332...
\]
We check that \( \phi^*(t_0) = \lambda_1 \).
3.3 The Range of Slopes

Figure 3.3 shows the (blue) set $\Omega$ of values $(b,t)$ satisfying $\phi(b,t) < \sqrt{3}$. These are the possible slopes of pairs of bends participating in a $T$-pattern with respect to a paper Mobius band of aspect ratio less than $\sqrt{3}$.

We have added in some points and lines to help frame $\Omega$, and in particular we have placed it inside a trapezoid which tightly hugs it. All the lines drawn touch $\partial \Omega$ except the red one, which lies just barely above $\Omega$. The lines of slope $2/3$ and $4/3$ through $(0,-1/\sqrt{3})$ are tangent to $\Omega$. The right vertex of $\Omega$ is $(a,-a/2)$ where $a = (\sqrt{27} - \sqrt{11})/4$.

One thing we notice right away is $(b,t) \in \Omega$ implies that $b > 0$ and $t < 0$. That is, the bends have opposite slopes. In particular, Figure 3.2 accurately depicts the situation. So, we have the following theorem:

**Theorem 3.2** If $\mathcal{M}$ is an immersed paper Mobius band with a $T$-pattern and aspect ratio less than $\sqrt{3}$ then $\mathcal{M}$ has at least 2 bends of slope 0.

**Proof:** This follows from our observation about the slopes together with the intermediate value theorem. ♠
More geometrically, the bends of slope 0 are perpendicular to the boundary of the Mobius band. Note that Theorem 3.2 applies to all embedded paper Mobius bands of aspect ratio less than $\sqrt{3}$, assuming that there are any. We have stated Theorem 3.2 in such a way that it is definitely not vacuous.

4 References

[FT], D. Fuchs, S. Tabachnikov, Mathematical Omnibus: Thirty Lectures on Classic Mathematics, AMS 2007