A Surface Group Can Act Properly on a Product of two Regular Trees

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Abstract

Let $T_p$ be the regular infinite $(p + 1)$-valent tree, also known as the Serre $p$-adic tree. Darren Long and Alan Reid made a spectacular guess as to a surface group $\Gamma$ that possibly could act properly on the product $X = T_2 \times T_3$. We verify that their guess works. It turns out that $X$ contains a subset $Y$ which is the proper image of a parallelogram-tiled infinite translation surface that is quasi-isometric to the hyperbolic plane $H^2$. The group $\Gamma$ acts on $Y$ in a way that is combinatorially identical to the way that $\Gamma$ acts on a certain tiling of $H^2$ by hyperbolic quadrilaterals.

1 Introduction

1.1 Some History

A group $\Gamma$ acts properly on a space $X$ if for every compact $K \subset X$, the set

$$\{ \gamma \in \Gamma | \gamma(K) \cap K \neq \emptyset \}$$

if finite. I learned about the following question from David Fisher when he visited Brown in early 2019: Can a surface group act properly on the product of infinite, finite valence, trees?

This is perhaps an old question, but in particular it is listed as Question 1 in the paper [FLSS], written by David Fisher, Michael Larsen, Ralf Spatzier,

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and Matthew Stover. Constructions in Teichmuller theory give examples of a surface group acting properly on the product of two $\mathbb{R}$-trees, but so far nobody has found a way to replace $\mathbb{R}$-trees by finite valence trees. It seems that people are divided as to whether this is possible.

David explained to me that Darren Long and Alan Reid have a candidate construction. Let $T_p$ be the infinite regular tree of valence $p + 1$. Serre’s famous book [S], called Trees, he describes how $GL_2(\mathbb{Q})$ naturally acts on $T_p$. The action is really defined for the larger group $GL_2(\mathbb{Q}_p)$ of nonsingular $2 \times 2$ matrices with $p$-adic entries. We will formally recall how this works in §4, and also explain how we compute the action in practice.

Up to scaling and $GL_2(\mathbb{Q})$-conjugation, Long and Reid’s group $\Gamma$ is generated by $a$ and $b$ where

$$a = \begin{bmatrix} 9 & 2 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & -16 \\ 4 & 83 \end{bmatrix}. \quad (1)$$

They conjecture that the Serre action of (a conjugate of) $\Gamma$ on $X = T_2 \times T_3$ is proper. The advantage $\Gamma$ has over the original group is that one of the commutators

$$c = a^{-1}bab^{-1} = \begin{bmatrix} -9 & -2 \\ 41 & 9 \end{bmatrix} \quad (2)$$

lies in $SL_2(\mathbb{Z})$ and therefore fixes the natural origin in $X$. This makes the orbit map $\gamma \to \gamma(O)$ especially attractive.

Though $\Gamma$ may not look like much at first, it is in reality a deep and beautiful guess. As Alan Reid explained to me, (their conjugate of) $\Gamma$ fits into a 2-parameter family of 2-generator groups which are orbifold fundamental groups of hyperbolic tori having one cone angle of $\pi$. The matrix entries are rational functions of the parameters. By cleverly choosing the parameters, Reid and Long controlled the prime factorization of the entries in a promising way. That is roughly how their guess arose.

Thanks to the nature of the hyperbolic quotient $H^2/\Gamma$, the commutator $c$ acts as an involution on the hyperbolic plane $H^2$, and $\Gamma$ preserves a tiling $\mathcal{T}$ by hyperbolic quadrilaterals having opposite sides of equal length. The $\Gamma$-orbit of the fixed point of $c$ is the vertex set of $\mathcal{T}$. The stabilizer subgroup of each vertex point has order 2. For the purposes of keeping this paper self-contained, I will sketch direct proofs of all these statements in §3.

Reid, Long, and others compiled much experimental evidence over the years that the guess works, but have not yet proved it.
1.2 Result and Proof Preview

The main purpose of this paper is to prove the following result.

**Theorem 1.1** The Serre action of $\Gamma$ on $X$ is proper.

Here is an outline of our proof. Guided by the combinatorics of $\mathcal{T}$ and our numerical experiments, we construct a space $Y_0$ which is a parallelogram-tiled nonpositively curved infinite translation surface. The parallelograms all have the same shape as the parallelogram $P \subset \mathbb{R}^2$ having vertices $\pm(3,1)$ and $\pm(3,3)$. In $Y_0$, there are 8 parallelograms fitting around a vertex, with the angles alternating acute and obtuse. All the cone angles are $4\pi$.

A group isomorphic to $\Gamma$ acts isometrically on $Y_0$ with compact quotient. We will produce a canonical equivariant quasi-isometry $\Phi : H^2 \to Y_0$ which maps tiles to parallelograms. Using the orbit map discussed above (and identifying $H^2$ with $Y_0$), we construct an equivariant and piecewise isometric mapping

$$\Psi : Y_0 \to X.$$  \hspace{1cm} (3)

$\Psi$ turns out to be an isometry in the complement of the vertices of $Y_0$. However, it is not a global isometric embedding, or even an embedding. Nonetheless, we prove the following result for all $a, b \in Y_0$.

**Lemma 1.2** If $\Psi(a) = \Psi(b)$ then $\text{distance}(a, b) \leq 2$.

**Proof of Theorem 1.1:** If $\Gamma$ does not act properly on $X$ then we can find an infinite sequence of words in $\Gamma$ which move the origin in $X$ a bounded distance. This gives an infinite number of vertices in $Y_0$ that $\Psi$ maps to a bounded neighborhood of the origin. By the pigeonhole principle, $\Psi$ must map two distant vertices to the same point, contradicting Lemma 1.2. ♠

**Remarks:**

(i) For a while I thought that I could prove that the map $\Psi$ was a quasi-isometry onto its image but my proof ran into trouble. I still wonder about this.

(ii) It is perhaps worth pointing out that $Y_0$ also has a description of a square-tiled translation surface, because it can alternately be built from a supply of $6 \times 2$ rectangles glued together 8 around each vertex. However, we find the parallelogram description more useful.
1.3 The Local Picture

We set \( Y = \Psi(Y_0) \). Here we explain why \( \Psi : Y_0 \to Y \) cannot be an embedding. If \( \Psi \) was an embedding, then the fact that \( \Psi \) is a local isometry away from vertices would force the link \( K_{3,4} \) of a vertex in \( X \) to contain an embedded 8-cycle. Since \( K_{3,4} \) only has 7 vertices, this is impossible. Hence \( \Psi \) cannot be an embedding, let alone an isometric embedding.

![Figure 1. An immersed 8-cycle in \( K_{3,4} \).](image)

At the same time, \( K_{3,4} \) does have an immersed 8-cycle which is homeomorphic to a figure 8 and invariant under an involution which sends \( (i) \) to \( (i + 4) \mod 8 \). Figure 1 shows an example. The 1-neighborhood of \( Y \) at each vertex \( v \in Y \) is isometric to a suitably metrized cone on such an 8-cycle. The action of the non-identity stabilizer of \( v \) in \( \Gamma \) implements the involution just mentioned.

1.4 Computer Assistance

I discovered the results in this paper using a graphical user interface I wrote in Java. The program draws (parts of) the trees. You can click on a vertex and see the representative lattice and also the action of a group element of your choice on that lattice. The program also draws part of the hyperbolic tiling, and you can click on the vertices of the tiling and see the corresponding elements in group \( \Gamma \) and their action on \( X \).

Our proof is also computer-assisted, though only mildly so. We need to perform about 32 calculations of how a certain 16 elements of \( \Gamma \) act on the
origins $O_2$ and $O_3$ of $T_2$ and $T_3$ respectively. One could do these by hand, but it would be painful and the chances of making a mistake would be large.

At the moment the program only works for $\Gamma$ and other subgroups of the group generated by

\[
\begin{bmatrix}
3 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
2 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
2 & 1 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
2 & 0 \\
1 & 2
\end{bmatrix},
\]

It also only deals with $T_2$ and $T_3$. I hope to make a more powerful version later on.

The program is available at

http://www.math.brown.edu/~res/Java/SERRE.tar

When you download the program you get a tarred directory. The directory (when untarred) has a README file with operating instructions.

1.5 Outline

Here is an outline of the paper. In §2, the conceptual part of the paper, I carry out all the steps above except some technical lemmas which rely on explicit calculations in the Serre trees. In §3, I explain in a direct way the main hyperbolic geometry features of $\Gamma$. In §4, I explain how I do the Serre tree calculations. In §5, I take care of the technical details by doing the calculations left over from §2. To be more precise, I should really say that I merely report on the calculations as they are done by my program. The interested reader can download the program and see every calculation.

1.6 Acknowledgements

I thank Nic Brody, David Fisher, Darren Long, Alan Reid, and Matthew Stover for helpful conversations related to this problem.
2 Proof modulo the Calculations

2.1 The Hyperbolic Tiling

Let $H^2$ denote the hyperbolic plane. Let $\Gamma = \langle a, b \rangle$ be the group given in Equation 1. Let $c = [A, b] = AbA^{-1}B$ be the commutator of $A$ and $B$. Here $A = a^{-1}$ and $B = b^{-1}$. Let $v_0 \in H^2$ be the fixed point of the involution $c$.

$\Gamma$ preserves a tiling $\mathcal{T}$ of $H^2$ by hyperbolic quadrilaterals having opposite sides of equal length and adjacent sides of unequal lengths. We call these quadrilaterals *tiles*. The vertex set of $\mathcal{T}$ is the $\Gamma$-orbit of $v_0$. Figure 2 shows a hand-drawn but accurate drawing of one flower of $\mathcal{T}$ in $H^2$. The numbers indicate vertices $v_j = w_j(v_0)$, where $w_1, ..., w_{16}$ are certain words we list in §3.2. (These numbers are unrelated to the numbers in Figure 1.)

![Figure 2. The hyperbolic tiling and a Euclidean parallelogram](image)

We color the edges of the tiling red and blue as shown in Figure 2. We also distinguish one diagonal in each tile. The pattern is that the distinguished diagonals in adjacent tiles do not share any vertices. When distinguished or undistinguished diagonals meet at a vertex, they meet at right angles.
2.2 The Infinite Translation Surface

Let $P$ be the Euclidean parallelogram having vertices $±(3, 1)$ and $±(3, 3)$. The right side of Figure 2 shows a Euclidean parallelogram having the same shape as $P$. We color the short sides of $P$ blue and the long sides red. We also distinguish the long diagonal of $P$. In other words, we decorate $P$ as in Figure 2. Let $Y_0$ be the space discussed in the introduction. Again, we build $Y_0$ by fitting 8 copies of $P$ around each vertex, so that the angles alternate acute and obtuse.

**Lemma 2.1** Up to an order 2 rotation, there is a unique homeomorphism from a hyperbolic tile $τ$ to $P$ which maps geodesic segments in $τ$ to line segments in $P$ and respects the decorations.

**Proof:** This proof only uses the fact that $τ$ is a hyperbolic quadrilateral whose opposite sides have the same length. We work in the Klein model, so that the geodesics are straight line segments and the isometry group acts by projective automorphisms of the unit disk. This means that under a suitable isometry, $τ$ is a Euclidean rectangle. In this case, there are two decoration respecting affine maps from $τ$ to $P$. These maps differ by an order 2 rotation, and they are the unique maps which do the job. ♠

Here is the quasi-isometric map $Φ : H^2 \to Y_0$. We first choose the homeomorphism from Lemma 2.1 and then we extend from tile to tile so as to make the whole map a homeomorphism. What makes $Φ$ fit together across tile edges is that the maps from Lemma 2.1 send the hyperbolic midpoints of the tiles to the midpoints of edges of the parallelograms. The map $Φ$ is a kind of combinatorial isomorphism in which we are just switching the geometry of the tiles.

**Remarks:**
(i) Even though we have taken the trouble to define this beautiful equivariant quasi-isometry, we don’t really use its nice properties. Any equivariant and tile-preserving mapping would work for us.
(ii) Even though $Y_0$ is quasi-isometric to $H^2$, the geodesics in $Y_0$ are rather different from those in $H^2$. We will always work with geodesics in $Y_0$, even though our pictures will show $H^2$. The space $Y_0$ is too hard to draw!
2.3 The Orbit Map

We identify the space $Y_0$ with $H^2$ in the way determined by the map $\Phi$ just discussed. Really, we are just changing the local geometry but keeping everything else – the combinatorics, decorations, and group action – the same. One could think of $Y_0$ as $H^2$ with a different $\Gamma$-invariant metric. Figure 2 will be our guide sometimes.

We first establish some notation. Recall that $X = T_2 \times T_3$ is the product two Serre trees, the 2-adic tree and the 3-adic tree. Let $\pi_p : X \rightarrow T_p$ be projection for $p = 2, 3$. As mentioned in the introduction, each matrix $M \in \Gamma$ acts on $T_2$ and $T_3$ by the Serre action.

Given a pair of points $\psi_1, \psi_2 \in X$ we associate a pair of integers $(d_2, d_3)$, where $d_p$ is the distance from $\pi_p(\psi_1)$ to $\pi_p(\psi_2)$ in $T_p$. We call $(d_2, d_3)$ the pitch of the pair $(\psi_1, \psi_2)$. Likewise we define the pitch of a geodesic segment in $X$ to be the pitch of its pair of endpoints. The usual metric on $X$ computes the distance between $\psi_1$ and $\psi_2$ as $\sqrt{d_2^2 + d_3^2}$.

Let $V$ denote the vertex set of $Y_0$. Let $v_0 \in V$ be the vertex fixed by $c$. The element $c \in \Gamma$ also fixes the origin $O \in X$, as we mentioned in the introduction. We define the orbit map $\Psi : V \rightarrow Y$:

$$\Psi(v) = \gamma(O), \quad v = \gamma(v_0).$$

The stabilizer of $v_0$ in $\Gamma$ is just the order 2 group generated by $c$. Thus $\Psi$ is a well-defined map.

Now we wish to extend $\Psi$ to the rest of $Y_0$. We first present some motivating calculations. Here, and below, the vertex $v_j$ corresponds to the number $j$ in Figure 2. We have $v_j = w_j(O)$ where $w_j$ is the $j$th word listed in §3.2. Let $\psi_j = \Psi(v_j)$. We compute the following for the vertices $v_0, v_1, v_2, v_3$ of a tile.

1. Blue edge: The pitch of $(\psi_0, \psi_1)$ is $(0, 2)$.
2. Blue edge: The pitch of $(\psi_2, \psi_3)$ is $(0, 2)$.
3. Red edge: The pitch of $(\psi_1, \psi_2)$ is $(6, 4)$.
4. Red edge: The pitch of $(\psi_3, \psi_0)$ is $(6, 4)$.
5. Distinguished diagonal: The pitch of $(\psi_1, \psi_3)$ is $(6, 6)$.
6. Undistinguished diagonal: The pitch of $(\psi_2, \psi_0)$ is $(6, 2)$.
In short, the pitches precisely calculate the displacement vectors of the corresponding edges of the parallelograms comprising $Y_0$. This, of course, is why we defined $Y_0$ this way. These calculations suggest the following result, which we also establish by direct calculation in §4.

**Lemma 2.2** The points $\psi_0, \psi_1, \psi_2, \psi_3$ are the vertices of a geodesically embedded Euclidean parallelogram which is isometric to a tile of $Y_0$.

Thus, we can extend our map $\Psi$ to all of $Y_0$ by making its restriction to each tile the kind of isometry which we just discussed. $\Psi$ sends the distinguished diagonals in each tile to a segment of pitch $(6, 6)$. The map sends the undistinguished diagonals to segments of pitch $(6, 2)$.

By construction, $\Psi$ is an isometric embedding when restricted to the interior of each tile. We also establish the following result, which actually subsumes Lemma 2.2.

**Lemma 2.3** Let $\tau_1$ and $\tau_2$ be two adjacent tiles of $Y_0$. The restriction of $\Psi$ to the interior of $\tau_1 \cup \tau_2$ is an isometric embedding.

Lemma 2.3 shows that $\Psi$ is a local isometric embedding of $Y_0 - V$ into $X$. In other words, the only place where $\Psi$ may not be a local isometry is at the vertices. We already explained in the introduction that $\Psi$ cannot be an isometric embedding.

### 2.4 Main Argument

Let $I$ be an interval. We say that $f : I \to T_p$ is monotone if the preimages of points are connected. A constant path counts as monotone. Geodesics in $X$ have monotone projections.

Let $\gamma$ be a geodesic in $Y_0$. Define

$$\gamma_p = \pi_p \circ \Psi(\gamma).$$

In §4 we prove the following result.

**Lemma 2.4** Let $\gamma$ be a small geodesic segment defined in the neighborhood of a vertex of $Y_0$ and containing that vertex. Then $\gamma_3$ is monotone.
Global Monotonicity: Lemma 2.4 has a nice consequence. Since $\Psi(\gamma)$ is a local geodesic when $\gamma$ does not contain a vertex of $Y_0$, we conclude that $\gamma_3$ is monotone provided that $\gamma$ does not contain a vertex of $Y_0$. Combining this with Lemma 2.4 we see that $\gamma_3$ monotone for any geodesic segment $\gamma$.

We call a direction in $Y_0$ horizontal if $\pi_3 \circ \Psi$ is stationary in this direction. The geodesic segment connecting $v_0$ to $b(v_0)$ is horizontal. Indeed, the geodesic connecting $O$ to $b(O)$ in $X$ has pitch $(6,0)$. See §5.5 for details. There are 4 horizontal geodesic segments in $Y_0$ emanating from $v_0$. They connect $v_0$ to the vertices corresponding to $w(b_0)$ where $w \in \{b, B, bc, Bc\}$. In §5.5 we prove the following easy lemma.

**Lemma 2.5** Given a pair of unequal words

$$(w_1, w_2) \in \{b, B, bc, Bc\}$$

the pair $(w_1(O), w_2(O))$ has pitch either $(12,0)$ or $(10,0)$.

Therefore, if $\gamma$ is the geodesic segment in $Y_0$ connecting $w_1(v_0)$ to $w_2(v_0)$ then $\gamma_2$ is either a geodesic or makes a single fold. It will emerge from our proof of Lemma 2.6 below that, in the fold case, the fold occurs right in the center of $\gamma_2$, as shown in Figure 3.
Lemma 2.6 If $\gamma$ is a horizontal geodesic segment of length greater than 2, then the endpoints of $\gamma$ are distinct.

Proof: Since $\Psi$ is a local isometry away from vertices and $\pi_3$ is trivial in the vertical direction, $\gamma_2$ moves at unit speed. This curve in $T_2$ is either monotone or makes single folds. The single folds are separated by at least 4 units. Therefore, the endpoints of $\gamma_2$ are distinct if $\gamma_2$ has length greater than 2. This implies our claim. ♠

Proof of Lemma 1.2: Suppose that $\Psi(a) = \Psi(b)$ for $a, b \in Y_0$. Since $Y_0$ is complete and has nonpositive curvature, we can join $a$ and $b$ by a unique geodesic segment $\gamma$. (We don’t actually use the uniqueness of the geodesic segment.) The segment $\gamma_3$ is monotone in all cases, so if $\Psi(a) = \Psi(b)$ the arc $\gamma_3$ must be stationary. This means that $\gamma$ is horizontal. Lemma 2.6 finishes the proof.
3 The Hyperbolic Picture

3.1 The Group
Recall that $\Gamma$ is generated by $a$ and $b$ from Equation 1 and $c = AbaB$. Again,

$$a = \begin{bmatrix} 9 & 2 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & -16 \\ 4 & 83 \end{bmatrix}, \quad c = \begin{bmatrix} -9 & -2 \\ 41 & 9 \end{bmatrix}$$ (6)

We let $\Gamma$ act on the upper half plane by the action:

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}(z) = \frac{\alpha z + \gamma}{\beta z + \delta}.$$ (7)

Note that this action is non-standard: It is the transpose of the usual action. The fixed point of $c$ is

$$z_0 = \frac{9}{2} + \frac{1}{2}i.$$ 

For Figure 2, we map $z_0$ to the origin of the unit disk using the map

$$T(w) = \frac{w' - i}{w' + i}, \quad w' = 2w - 9.$$ 

3.2 The Tiling
For convenience, we repeat Figure 2. Here are the words we mentioned in §2.1.

1. $w_1 = A$.
2. $w_2 = ba$.
3. $w_3 = baa$.
4. $w_4 = baaa$.
5. $w_5 = a$.
6. $w_6 = ABc$.
7. $w_7 = AABc$.
8. $w_8 = AABcA.$
The other words are determined by the rule that $w_{8+j} = w_{j}c$. In terms of group actions, our notation is such that $d_1\ldots d_n$ means “First apply $d_1$, then $d_2$, etc.”

![Figure 4. The hyperbolic tiling](image)

### 3.3 The Geometry of the Group

For the sake of writing a self-contained paper, we justify the various geometrical claims about $\Gamma$ by direct calculation.

The vertex $v_0$ in the Poincare model corresponding to the word $w_1 = A$ is given by

$$T \circ A(z_0) = \frac{9}{10} + \frac{i}{10}.$$  

This looks about right in the picture. In the upper half plane, the corresponding point is

$$z_1 = A(z_0) = \frac{-1}{2} + \frac{9}{2}i.$$
We also compute
\[ z_2 = (b.a)(z_0) = \frac{684}{425} + \frac{288}{425}i, \]
\[ z_3 = (b.a.a)(z_0) = \frac{29484}{8345} + \frac{2592}{8345}i. \]

The points \( z_0, z_1, z_2, z_3 \) are the vertices of one of the yellow quadrilateral \( \tau \) in Figure 4.

We can use these calculations to get the information we have used about the hyperbolic action of \( \Gamma \).

**Lemma 3.1** The opposite sides of \( \tau \) have the same length. These sides are paired by the elements \( A \) and \( baa \).

**Proof:** We compute that \( A(z_3) = z_2 \). Hence \( A \) pairs the red sides of \( \tau \) and they have the same length. We likewise compute that \( baa(z_1) = z_2 \). Hence \( baa \) pairs the blue sides of \( \tau \) and they have the same length. ♠

**Lemma 3.2** \( \Gamma \) acts properly on \( \mathbb{H}^2 \) and \( \tau \) is a fundamental domain for the action. Moreover, the stabilizer of \( z_0 \) is the order 2 subgroup consisting of \( c \) and the identity.

**Proof:** (sketch) Note that \( A \) and \( baa \) pair the opposite sides of \( \tau \). We compute that \( [A, baa] = c \). This fact implies that 8 translates of \( \tau \) fit around each vertex of \( \tau \). Finally \( A \) and \( baa \) generate \( \Gamma \). The Poincare Polyhedron Theorem now gives the desired results. ♠

**Lemma 3.3** The red and blue edges in the tiling fit together to form a union of geodesics.

**Proof:** This follows from the Poincare Theorem as well, but we can also do a direct calculation. We compute that \( v_5 = iv_1 \) and \( v_7 = iv_3 \). Therefore, around a vertex, consecutive red or blue edges form a right angle. This suffices. ♠
4 Calculating the Serre Action

4.1 Formal Definition of the Serre Action

Let $Q_p$ denote the $p$-adic field and let $Z_p$ denote the ring of $p$-adic integers. A rank two $Z_p$ module is a subset of $(Q_p)^2$ which is invariant under addition and $Z_p$ scaling. Two such modules $\Lambda_1$ and $\Lambda_2$ are declared equivalent if there is some $r \in Q_p$ such that $r\Lambda_1 = \Lambda_2$, and adjacent if there is some $r \in Q_p$ such that $r\Lambda_1 \subset \Lambda_2$ and $\Lambda_2/(r\Lambda_1)$ is isomorphic to $Z/p$.

Serre forms a graph whose vertices are equivalence classes of rank 2 $Z_p$ modules and joins them by an edge if two equivalence classes have adjacent representatives. Serre then proves that the resulting graph is the infinite regular tree of valence $T_p$. The group $GL_2(Q_p)$ acts on these rank two modules and preserves both the relations of equivalence and adjacency. This gives the Serre action on $T_p$.

4.2 An Integer Model for Serre Trees

I found these amazing results very hard to understand until I started thinking about it in terms of something much easier to understand – the connection between $Z_p$ and a rooted $p+1$-valent tree. This tree has one vertex of degree $p$ – the root – and the remaining vertices have degree $p + 1$.

Here is the description. $Z_p$ is the inverse limit of the chain of homomorphisms

$$\cdots \rightarrow Z/p^3 \rightarrow Z/p^2 \rightarrow Z/p^1 \rightarrow Z/p^0.$$  

Here $Z/p^0$ is just defined to be $\{0\}$. We form a tree by joining each element in $Z/p^k$ to the element in $Z/p^{k-1}$ to which it reduces. This gives us our rooted $p + 1$ valent regular tree. The ring $Z_p$ can be identified with the set of infinite paths in the tree. It turns out that we can make the Serre tree $T_p$ look very much like this much simpler object. In particular, we can take all the representatives to have an integer basis. We will first deal with the cases $p = 2$ and then consider the case $p = 3$ somewhat more briefly.

The case $p = 2$: We can take representatives for these modules which are certain lower triangular integer matrices, in which the diagonal entries are powers of 2. The module is the $Z_p$ span of the rows. We build the tree in layers, with the $n$th layer consisting of certain such matrices of determinant $2^n$. The left two-thirds of the tree has precisely the structure coming from our rooted tree. The right one-third of the tree is a kind of inversion of
the left two-thirds. Thus, we really just have to think about the rooted tree
construction and remember how the inversion works. Beautifully, each lattice
manifestly has index 2 in the one it joins to on the layer above. So, one can
see by inspection that this really is a correct model for $T_2$. Figure 5 shows
the first 4 layers of the Serre Tree $T_2$.

$$
\begin{array}{cccc}
1 & 0 \\
0 & 1 \\
2 & 0 & 2 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 2 \\
4 & 0 & 4 & 0 & 4 & 0 & 2 & 0 & 1 & 0 \\
0 & 1 & 2 & 1 & 1 & 1 & 3 & 1 & 1 & 2 & 0 & 4 \\
8 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 4 & 0 & 4 & 0 & 2 & 0 & 1 & 0 \\
0 & 1 & 4 & 1 & 2 & 1 & 6 & 1 & 1 & 1 & 5 & 1 & 3 & 1 & 7 & 1 & 3 & 2 & 1 & 2 & 4 & 0 & 8 \\
\end{array}
$$

Figure 5. The beginning of $T_2$.

$T_3$ works the same way, but with powers of 3 replacing powers of 2. Figure
6 shows the first 3 layers of $T_3$.

$$
\begin{array}{cccc}
1 & 0 \\
0 & 1 \\
3 & 0 & 3 & 0 & 3 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 2 & 0 & 3 \\
9 & 0 & 9 & 0 & 9 & 0 & 9 & 0 & 9 & 0 & 9 & 0 & 9 & 0 & 9 & 0 & 3 & 0 & 3 & 0 & 1 & 0 \\
0 & 1 & 3 & 1 & 6 & 1 & 1 & 1 & 4 & 1 & 7 & 1 & 2 & 1 & 5 & 1 & 8 & 1 & 2 & 3 & 1 & 3 & 0 & 9 \\
\end{array}
$$

Figure 6. The beginning of $T_3$.  

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4.3 Computing the Serre Action in Principle

In principle, we compute the Serre action on our model just by taking the relevant matrices, multiplying them out, and then matching the answer to the representative in our model. Even though we don’t really compute things this way, we include an example to illustrate our convention for multiplying matrices. Let

\[ M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \]

be our group element and let

\[ \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \]

be a lattice in our model. We compute

\[ M(\Lambda) = \Lambda \times M = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} \]

This is not in our model, but note that the module in question is the span of \( U = (2, 1) \) and \( V = (0, 4) \). This is also the span of \( U \) and \( 4U - V = (8, 0) \). Thus, in terms of the Serre action,

\[ M(\Lambda) = \begin{bmatrix} 8 & 0 \\ 2 & 1 \end{bmatrix} . \]

This one is in our model.

Remarks:
(i) To make the method here more in line with the hyperbolic geometry action, we could instead take the left transpose action

\[ M(\Lambda) = (M^t \times \Lambda^t)^t . \]

I somewhat regret not using more standard conventions but I consider it a miracle that my program actually works. At some point I will try to line up my conventions with whatever is in the literature.
(ii) When we are trying to figure out the equivalence, we never need to deal with \( \mathbb{Z}_p \). Instead we can work with the ring \( R_p \) consisting of fractions of the form \( a/b \) where \( b \) is relatively prime to \( p \). In other words, in our calculation above, we are free to take integer combinations of the form \( uU + vV \) with \( u, v \in R_p \). We can also scale by elements of \( R_p \).
4.4 Computing the Serre Action in Practice

I gave the example above mainly to explain how the action works in principle. For large words, it seems really quite difficult to find the equivalence. Sometimes, though not for the example above, we really need to work with the ring $R_p$. Instead of doing this, I found two tricks for doing and certifying the calculations. One trick is to simply write the generators of $\Gamma$ in terms of shorter generators and compute with these instead. The second trick is somewhat akin to Hensel’s lemma. I will explain these tricks in turn. My code implements these tricks.

**Simpler Generators:** We introduce the simpler matrices

\[
\mu_0 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mu_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mu_4 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mu_6 = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}.
\] (8)

and set $\mu_{j+1} = \mu_j^{-1}$ for $j = 0, 2, 4, 6$. These matrices do not belong to $\Gamma$ but we can use them to generate the generators $a$ and $b$ of $\Gamma$. From trial and error, I found that

\[
a = (35300242), \quad b = (3536633344444444433242).
\] (9)

This notation means that $a = \mu_3\mu_5\mu_3\mu_0...$ and likewise for $b$.

**Guided Lattice Finding:** Even computing with the simpler matrices is tough. Here is our trick. I think of this trick as being similar to Hensel’s Lemma, at least in spirit. We explain the trick for $T_2$. It works essentially the same way for $T_3$. We define the full reduction of a lattice $\Lambda$ in our model to be the sequence $\Lambda_0, ..., \Lambda_n = \Lambda$ comprising the path connecting the level 0 lattice to $\Lambda$. Let $\theta$ be one of our 8 matrices above. We first precompute $\theta(\Lambda_0)$. This is easy.

Next, assuming that we have computed $\theta(\Lambda_k)$, we know that $A = \theta(\Lambda_{k+1})$ must be one of the 3 lattices $B_1, B_2, B_3$ adjacent to $\theta(\Lambda_k)$. So, we only need to find one of three lattices and certify equivalence. To do this, we apply an auxiliary matching routine which finds a sufficient condition for $A$ and some $B_j$ to match. There is no theoretical reason to think that my matching routine always works, but it did work for about a million randomly chosen words in $\Gamma$ having length about 25. The matching routine breaks when we try it with some other smallish matrices, but we don’t care.
**Simple Matching:** First we explain a method which sometimes certifies that lattices \( E \) and \( F \) match. Let \( M^* \) denote the integer matrix such that \( MM^* = \det(M) \).

- We divide out by the gcd of the entries of \( E \) to get a new matrix \( E' \).
- We divide out by the gcd of the entries of \( F \) to get a new matrix \( F' \).
- We let \( G = (E')^*F' \).
- We check whether \( G' \in \pm SL_2(\mathbb{Z}) \). If so, then \( E \) and \( F \) represent the same point in the Serre tree.

This method is not very powerful, so we augment it with something better.

**Enhanced Matching:** Our enhanced matching method is pretty *ad hoc*. The approach is to construct several lattices equivalent to \( E \) and then try to match one of these with \( F \) using the simple matching algorithm. First of all, given a vector \( U \in \mathbb{Z}^2 \) we let \( U^* \in \mathbb{Z}^2 \) denote the vector \( U/u \) where \( u \) is largest common odd divisor of the components of \( U \). For instance, if \( U = (12, 18) \) then \( u = 3 \) and \( U^* = (4, 6) \). Let \( E \) be the span of \( U \) and \( V \).

- \( E_1 \) is the span of \( U^* \) and \( V^* \). Call these vectors \( U_1 \) and \( U_2 \).
- \( E_2 \) is the span of \( U_1 \) and \((U_1 + V_1)^* \).
- \( E_3 \) is the span of \( U_1 \) and \((U_1 - V_1)^* \).

The matrices \( E_0, E_1, E_2 \) all represent the same point in the Serre tree. We try to match \( F \) with each of these. The method succeeds if we get at least one match. Like the simple method, this method is not guaranteed to work. However, it does all our calculations for us just fine.
\section{The Technical Lemmas}

\subsection{Embedding a Single Tile}

Here we prove Lemma 2.2. We denote geodesic segments in $T_2$ and $T_3$ by their endpoints. Such segments will be denoted by square brackets. Thus $[E, F]_2$ will represent a geodesic segment in $T_2$. The words corresponding to the 4 vertices of the quadrilateral $\tau$ discussed above are

- $w_0 = \emptyset$
- $w_1 = A$
- $w_2 = ba$
- $w_3 = baa$

Here $w_0$ is the empty word. This word acts as the identity on the space $X$. Let $\Lambda_0$ denote the level 0 lattice in the Serre tree.

\begin{equation}
\begin{split}
  w_0(\Lambda_0) &= \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right). \quad (10) \\
  w_1(\Lambda_0) &= \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 4 & 1 \end{bmatrix} \right). \quad (11) \\
  w_2(\Lambda_0) &= \left( \begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \right). \quad (12) \\
  w_3(\Lambda_0) &= \left( \begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix}, \begin{bmatrix} 81 & 0 \\ 0 & 1 \end{bmatrix} \right). \quad (13)
\end{split}
\end{equation}

These calculations give us the pitches advertised in §2.3. In $X$, all 4 vertices lie in the product

\begin{equation}
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix} \times \begin{bmatrix} 9 & 0 \\ 4 & 1 \end{bmatrix}, \quad \begin{bmatrix} 81 & 0 \\ 0 & 1 \end{bmatrix} \end{equation}

This is a $6 \times 6$ flat Euclidean rectangle in $X$. This proves Lemma 2.2 for the quadrilateral $\tau$. Given that $\tau$ is a fundamental domain for the action of $\Gamma$ on $H^2$, the action of $\Gamma$ on $X$ transitively permutes the sets of interest to us. Hence, by symmetry, Lemma 2.2 holds for all of the parallelogram tiles. To be sure, we also calculated the pitches for all 8 quadrilaterals in Figure 5.
5.2 Embedding Adjacent Tiles

Here we prove Lemma 2.3. Let $\tau_2$ be the tile adjacent to $\tau_1 = \tau$ across the red diagonal edge. The vertices of $\tau_2$ are $v_0, v_3, v_4, v_5$. We compute

$$w_4(\Lambda_0) = \left( \begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix}, \begin{bmatrix} 729 & 0 \\ 0 & 1 \end{bmatrix} \right). \quad \quad w_5(\Lambda_0) = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

(15)

This time all the vertices of $\Psi(\tau_1 \cup \tau_2)$ are contained in the product

$$\left[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix} \right]_2 \times \left[ \begin{bmatrix} 9 & 0 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 729 & 0 \\ 0 & 1 \end{bmatrix} \right]_3.$$

(16)

This is a flat $6 \times 8$ rectangle. The two flat rectangles $\Psi(\tau_1)$ and $\Psi(\tau_2)$ share an edge and have the same shape. They either coincide or lie on opposite sides of the same edge. Since they don't coincide, they lie on opposite sides of the same edge. Hence $\Psi(\tau_1 \cup \tau_2)$ is another flat parallelogram. By symmetry, Lemma 2.3 holds for all tiles which are adjacent across a red edge.

Let $\tau_3$ be the tile in Figure 5 that is adjacent to $\tau_1$ across a blue edge. The vertices of $\tau_3$ are $v_0, v_{15}, v_{16}, v_1$. We have

$$w_1(\Lambda_0) = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 4 & 1 \end{bmatrix} \right).$$

(17)

$$w_2(\Lambda_0) = \left( \begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

(18)

$$w_3(\Lambda_0) = \left( \begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix}, \begin{bmatrix} 81 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

(19)

$$w_{15}(\Lambda_0) = \left( \begin{bmatrix} 4 & 0 \\ 3 & 16 \end{bmatrix}, \begin{bmatrix} 81 & 0 \\ 76 & 1 \end{bmatrix} \right).$$

(20)

$$w_{16}(\Lambda_0) = \left( \begin{bmatrix} 4 & 0 \\ 3 & 16 \end{bmatrix}, \begin{bmatrix} 729 & 0 \\ 643 & 1 \end{bmatrix} \right).$$

(21)

All these vertices lie in the product

$$\left[ \begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 3 & 16 \end{bmatrix} \right]_2 \times \left[ \begin{bmatrix} 729 & 0 \\ 643 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 4 & 1 \end{bmatrix} \right]_3.$$

(22)

This works because $643 \equiv 76$ mod 81 and $76 \equiv 4$ mod 9. The same argument as in the red case finishes the proof. This completes the proof of Lemma 2.3.
5.3 Relevant Indices

Before proving Lemma 2.4, we discuss the nature of geodesics in $Y_0$ which hit vertices. The space $Y_0$ is isometric to a Euclidean cone in the neighborhood of a vertex. Figure 7 shows (in the hyperbolic model) how we triangulate this cone. The restriction of $\Psi$ to each triangle is an isometric embedding. A geodesic which starts in the yellow triangle cannot exit in the grey region, because then it would make an angle of less than $\pi$ at the cone point. It has to “skip three” triangles on either side before exiting.

![Figure 7. The skip three rule](image)

The skip three rule is necessary but not sufficient. Some of the piecewise geodesics which enter in the yellow triangle and exit in a red triangle are geodesics and some are not. We will have to examine these cases carefully when the time comes.

When $(i, j)$ are such that $j \in \{i+5, \ldots, i+11\}$ we call $(i, j)$ a *sweetspot*. In this case, any piecewise geodesic starting in $\tau_i$, ending in $\tau_j$, and containing $v_0$ is a geodesic. Here $\tau_i$ is the triangle with vertices $(v_0, v_i, v_{i+1})$. We call the pairs $(i, i+4)$ and $(i, i+12)$ *fringe*. These are the pairs where we have to look more carefully at the situation.

We call a pair $(i, j)$ *relevant* if it is either sweetspot or fringe.
5.4 The Monotone Projection

Here we prove Lemma 2.4. We just have to analyze geodesics in the flower shown in Figure 8 which go through the central vertex. We first list the 16 lattices of $T_3$ corresponding to the vertices in Figure 8. We have computed some of these already, but here we gather them all together.

$\Lambda_1 = \begin{bmatrix} 9 & 0 \\ 4 & 1 \end{bmatrix} \quad \Lambda_2 = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \quad \Lambda_3 = \begin{bmatrix} 81 & 0 \\ 0 & 1 \end{bmatrix} \quad \Lambda_4 = \begin{bmatrix} 729 & 0 \\ 0 & 1 \end{bmatrix} \quad \Lambda_5 = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \quad \Lambda_6 = \begin{bmatrix} 9 & 0 \\ 5 & 1 \end{bmatrix} \quad \Lambda_7 = \begin{bmatrix} 81 & 0 \\ 77 & 1 \end{bmatrix} \quad \Lambda_8 = \begin{bmatrix} 729 & 0 \\ 77 & 1 \end{bmatrix} \quad \Lambda_9 = \begin{bmatrix} 9 & 0 \\ 5 & 1 \end{bmatrix} \quad \Lambda_{10} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \quad \Lambda_{11} = \begin{bmatrix} 9 & 0 \\ 5 & 9 \end{bmatrix} \quad \Lambda_{12} = \begin{bmatrix} 81 & 0 \\ 41 & 9 \end{bmatrix} \quad \Lambda_{13} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \quad \Lambda_{14} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \quad \Lambda_{15} = \begin{bmatrix} 9 & 0 \\ 4 & 1 \end{bmatrix} \quad \Lambda_{16} = \begin{bmatrix} 81 & 0 \\ 76 & 1 \end{bmatrix}$

We take indices mod 16. Recall that triangle $\tau_i$ has vertices $(v_0, v_i, v_{i+1})$. We associate to $\tau_i$ the convex hull of of the 3 lattices in $T_3$ associated to these vertices. We call this set $H(i)$. The set $H(i)$ is just the union of the full reductions of the lattices associated to $L_i$ and $L_{i+1}$. (We discussed this concept in §4.4, in connection with guided lattice finding.) Here are 4 examples:

$H(1) = \begin{bmatrix} 9 & 0 \\ 4 & 1 \\ 3 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 3 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

$H(5) = \begin{bmatrix} 9 & 0 \\ 0 & 1 \\ 3 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 3 & 0 \\ 2 & 1 \\ 5 & 1 \end{bmatrix}$

$H(6) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 9 & 0 \\ 2 & 1 \\ 3 & 0 \\ 5 & 1 \\ 27 & 0 \\ 23 & 1 \\ 77 & 1 \end{bmatrix}$

$H(9) = \begin{bmatrix} 9 & 0 \\ 5 & 1 \\ 3 & 0 \\ 2 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 3 \\ 0 & 9 \end{bmatrix}$

Let us compare $H(1)$ and $H(6)$. Notice that they only have one lattice in common. Suppose that $\gamma \subset \tau_1 \cup \tau_6$ is a geodesic segment as in Lemma 2.4. We want to see that $\gamma_3$ is monotone. If not, then $\gamma_3$ has some folding. This means that there are points $x_1, y_1 \in \gamma \cap \tau_1$ and $x_6, y_6 \in \gamma \cap \tau_6$ such that
But all these images are contained in the two relevant convex hulls, and the situation above is only possible if $H(1)$ and $H(6)$ have more than one lattice in common. Since they do not, we get a contradiction. Hence $\gamma_3$ is monotone. This takes care of one case of Lemma 2.4.

Our computer program checks, for each of the $16 \times 7$ sweetspot pairs $(i, j)$, that $H(i)$ and $H(j)$ only have one lattice in common. This establishes Lemma 2.4 for all sweetspot pairs.

For the fringe pairs, we check that the two convex hulls share just one vertex in common except for the pairs $(1, 5), (5, 9), (9, 13), (13, 1)$. Let us look carefully at $H(1)$ and $H(5)$. We orient $H(1)$ so that the initial point is the lattice associated to $v_1$ and the final point is the lattice associated to $v_2$. We orient $H(5)$ so that the initial point is the lattice associated to $v_5$ and the final point is the lattice associated to $v_6$.

$H(1)$ and $H(5)$ have 3 vertices in common, but it is better to say that they have 2 edges in common. More precisely, the last two edges of $H(1)$ coincide with the first two edges of $H(5)$. For this reason, if $\gamma$ is an actual geodesic segment in $\tau_1 \cup \tau_5$, then $\pi_3 \circ \Psi$ maps the endpoints of $\gamma$ to non-overlapping parts of the two hulls. Therefore, $\gamma_3$ is monotone. To be sure, we mention that the lattices associated to $v_1$ and $v_5$ are 4 units apart, so we can see directly that $\Psi(\gamma)$ is a geodesic segment when $\gamma$ connects $v_1$ to $v_5$. The same goes when $\gamma$ connects $v_2$ to $v_6$.

The same analysis works for $H(5)$ and $H(9)$. The other two pairs follow from symmetry, via the action of the commutator $c$ which fixes $v_0$ and rotates by $\pi$.

This completes the proof of Lemma 2.4.

5.5 The Horizontal Directions

Here we prove Lemma 2.5. Since $b$ and $c$ both fix $O_3$, we can compute the desired pitch of the segments of interest to us just by looking at the picture in $T_2$. 

24
We first list the lattices in $T_2$ corresponding to the words in the word list $\mathcal{H} = \{b, B, bc, Bc\}$. They are respectively
\[
\begin{bmatrix} 64 & 0 \\ 44 & 1 \end{bmatrix}, \quad \begin{bmatrix} 4 & 0 \\ 3 & 16 \end{bmatrix}, \quad \begin{bmatrix} 64 & 0 \\ 45 & 1 \end{bmatrix}, \quad \begin{bmatrix} 32 & 0 \\ 17 & 2 \end{bmatrix}.
\] (23)

For each $w \in \mathcal{H}$ the segment joining $O_2$ to $w(O_2)$ in $T_2$ has length 6. The lattices on these paths, at distance 2 from $O_2$, are respectively
\[
\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}.
\] (24)

Since these lattices are all distinct, we see that the $T_2$-distance from $w_i(O_2)$ to $w_j(O_2)$, for all unequal $i, j$, is at least 10. This completes the proof of Lemma 2.5.
6 References
