(Euclidean)

DIMERS AND GEOMETRY

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Based on work with

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1. Spaces of t-graphs and planar graph embeddings.

2. Tilings with convex polygons of fixed slopes.

3. Integrable systems of t-graphs
Thm[Kasteleyn(1965)]:

$$\det K = \sum_{\text{dimer covers } m} wt(m).$$

Q. What is the geometry underlying $K$?
A \textit{t-graph in a polygon} is a union of noncrossing line segments in which every endpoint lies on another segment or the boundary.

A t-graph is \textit{generic} if no two endpoints are equal.

For generic t-graphs,

\[ 1 = \chi(\text{open disk}) = \#(\text{faces}) - \#(\text{segments}). \]
local pictures:

- generic
- nongeneric
- generic
- nongeneric
- generic
- nongeneric
- generic
- nongeneric
- generic
Associated to a t-graph is a bipartite graph...
...which has dimer covers (when we remove all but one outer edge).
Thm: The space of t-graphs with $n$ segments, fixed boundary and fixed combinatorics is homeomorphic to $\mathbb{R}^{2n}$.

Global coordinates are *biratio coordinates* $\{X_i\}$.

Proof ideas: linear algebra and the maximum principle. □
At a degenerate vertex, biratios are defined by continuity:

\[ X = \frac{c \sin \theta_3}{a \sin \theta_2} \quad Y = \frac{a \sin \theta_1}{b \sin \theta_3} \quad Z = \frac{b \sin \theta_2}{c \sin \theta_1} \]

Note \( XYZ = 1 \).
Special case 1. Convex embeddings of graphs

An embedding of a graph in $\mathbb{R}^2$ is *convex* if its faces are convex

**Thm:** The space of convex embeddings of $G$ (with pinned boundary) is homeomorphic to $\mathbb{R}^{2V}$. 

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Proof: Take a nearby nonegenerate t-graph and set products of biratios around “vertices” to be 1. Show that any such assignment of biratios results in an embedding. □
Special case 2.

Product of $X$s around both faces and vertices is 1.

$\implies$ harmonic embeddings

Spring networks / Resistor networks
Random convex embedding  Random harmonic embedding
A random convex embedding does not have a scaling limit shape.
Special case 3. discrete analytic functions  (Fix exact shapes up to scale)  
e.g. regular hexagons and equilateral triangles (all $X$’s equal to 1.)
Rectangle tilings

(square young tableau limit shape)
Part 2. T-graphs with fixed slopes
Polygons (closed polygonal curves) with fixed edge slopes

**Thm[Thurston]:** Given a convex $n$-gon, the space of closed polygonal curves with the same edge slopes is $\mathbb{R}^{n-2}$. On this space the area is a quadratic form of signature $(1, n - 3)$.

Proof by picture:

Thus for fixed area there are two components, called *orientations*.
Take a t-graph in polygon with fixed combinatorics and edge slopes.

**Thm:** For fixed generic slopes and fixed areas, there is exactly one (generalized) tiling for each orientation.

**Conjecture [Reality]:** Solutions are in a totally real extension field of \( \mathbb{Q}[\text{slopes, areas}] \).
Thm: For each choice of orientation, the set of possible areas (if nonempty) is homeomorphic to a closed ball of dimension $F$.

Proof: The map $\Psi : \{\text{intercepts}\} \rightarrow \{\text{areas}\}$ is a local homeomorphism because $D\Psi$ is a Kasteleyn matrix for the underlying bipartite graph. Injectivity of $\Psi$ follows from convexity: given two tilings with same areas and same orientations, their average has greater area for each tile. $\square$
For a rectangle tiling, *all* areas possible (but not all orientations...)

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Conclusion:

For any weighted planar bipartite graph, $K = D\Psi$

$K$ is the differential of a (geometrically defined) map.
Q. What are natural probability measures on the space of t-graphs with fixed combinatorics?
Convex embeddings on the torus
you have been watching

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thank you for your attention!