LINEAR RESOLVENT GROWTH OF RANK ONE PERTURBATION OF A UNITARY OPERATOR DOES NOT IMPLY ITS SIMILARITY TO A NORMAL OPERATOR.

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To the memory of Tom Wolff

ABSTRACT. The main result of this paper is that for any unitary (self-adjoint) operator U with non-trivial absolutely continuous part of the spectrum there exists a rank one perturbation $K = ba^* = (\,\cdot\,, a)b$, such, that the operator $T = U + K$ satisfies the Linear Resolvent Growth condition (LRG),

$$\|(\lambda I - T)^{-1}\| \le C / \operatorname{dist}(\lambda, \sigma(T)), \qquad \lambda \in \mathbb{C} \setminus \sigma(T),$$

its spectrum lies on the unit circle $\mathbb{T}$ (on the real line $\mathbb{R}$), but T is not similar to a normal operator.

This contrasts sharply with the result of M. Benamara and the first author [1] that if a finite rank perturbation $T = U + K$ of a unitary operator is a *contraction* ($\|T\| \le 1$), then it is similar to a normal operator if and only if it satisfies (LRG) and its spectrum does not cover the unit disc $\mathbb{D}$.

NOTATION

$\mathbb{D}$	Open unit disk in the complex plane $\mathbb{C}$, $\mathbb{D} := \{z \in \mathbb{C} \,:\, |z| < 1\}$;
$\mathbb{T}$	Unit circle, $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| = 1\}$;
$\mathbb{C}_+$	Upper half-plane, $\mathbb{C}_+ := \{z \in \mathbb{C} \,:\, \operatorname{Im} z > 0\}$;
R_λ^T	Resolvent of the operator T, $R_\lambda^T := (\lambda I - T)^{-1}$;
a^*	If a is a vector in a Hilbert space, then $a^* := (\,\cdot\,, a)$ stands for the linear functional (operator) $f \mapsto (f, a)$;
ba^*	Stands for the operator $ba^* := b(\,\cdot\,, a)$, i. e. for the operator $f \mapsto (f, a)b$;
H^2, H_-^2	Hardy class of analytic and antianalytic functions in the disk or in the upper half-plane;
P_+, P_-	Riesz projections i. e. orthogonal projections onto H^2 and H_-^2 respectively;
m	Normalized ($m(\mathbb{T}) = 1$) Lebesgue measure on $\mathbb{T}$;

$|\mathcal{I}|$ Length of the arc (interval) $\mathcal{I}$;

0. Introduction

The similarity problem consists in finding criteria for a given operator T in a Hilbert space H to be similar to a normal operator. Recall, that an operator N in a Hilbert space H is called normal if $N^*N = NN^*$ and the operators N and T are similar if there exists a linear isomorphism V such, that $T = VNV^{-1}$. As it is well known, normal operators are characterized by the existence of an orthogonal spectral resolution, or by the spectral theorem—an infinitedimensional analogue of diagonalization of hermitian matrices (see below for the von Neuman version of the spectral theorem). This characterization is the main reason why the similarity is so important: being similar to a normal operator, the given operator allows an unconditionally convergent diagonalization (instead of the orthogonal one), which is enough for most consumers of spectral theory.

The most practically interesting case arises in the perturbation theory, meaning that the operator T is a "small" perturbation of a given normal operator N_0, $T = N_0 + K$. Classical cases are concerned with selfadjoint or unitary N_0. It is also worth mentioning, that that, in a sense, the classical scattering theory concerns in part with similarity problem. One of the main results, the famous Kato–Rosenblum theorem states, that for a selfadjoint operator N_0 and its perturbation $N_0 + K$ by a selfadjoint trace class operator K, the absolutely continuos parts of both operators are similar (and so unitarily equivalent) to each other.

For details on the long history of the problem we refer, for example, to [1], or to [11]. It is also worth mentioning that various approaches used in the field can be roughly classified as those based on Friedrichs' Γ-equations, or on J. Wermer's calculus, or on the Sz.-Nagy–Foias model representation.

In this paper, the reader can recognize some elements of the calculus approach. Namely, the basic Wermer's theorem [12] says that an operator T is similar to a normal one if and only if there exists a bounded homomorphism $\mathbf{j} : C(\sigma) \to L(H)$ from the algebra $C(\sigma)$ of all continuos functions on the spectrum $\sigma = \sigma(T)$ to the algebra $L(H)$ of all bounded linear operators on H, such that $\mathbf{j}(z) = T$; here z is the identity mapping of the complex plane, $z(\xi) \equiv \xi$ (independent variable). An immediate consequence is that an operator T similar to a normal one satisfies the following Linear Resolvent Growth condition

$$(\mathrm{LRG}) \qquad \|R_\lambda^T\| = \|(\lambda I - T)^{-1}\| \le \frac{C}{\mathrm{dist}(\lambda, \sigma(T))}, \qquad \lambda \in \mathbb{C} \setminus \sigma(T).$$

Of course, this observation does not depend on any theory and is a straightforward corollary of the similarity by itself: if $T = VNV^{-1}$, then

$$\|R_\lambda^T\| = \|VR_\lambda^N V^{-1}\| \le \|V\| \cdot \|V^{-1}\| \cdot \|R_\lambda^N\| = \|V\| \cdot \|V^{-1}\| \cdot \frac{1}{\mathrm{dist}(\lambda, \sigma(N))},$$

and $\sigma(T) = \sigma(N)$.

The simplest, and often the most practical test for the similarity would be the converse of the last observation. Thus, the Resolvent test Problem, as a part of Similarity Problem, is to know for what classes of T or of perturbations $T = N + K$ of a "good" normal operator N, the above LRG condition implies that T is similar to a normal operator. Such kind of results are important for applications, since LRG condition is usually easy to check.

Of course, the Linear Resolvent Growth is not sufficient for the similarity to a normal operator. The simplest example here is probably the *forward shift* S, i. e. the multiplication by the independent variable z in the Hardy space $H^2 = H^2(\mathbb{D})$, $Sf(z) = zf(z)$ (equivalent definition: $Se_n = e_{n+1}$, $n \geq 0$, where $\{e_n\}_{n=0}^{\infty}$ is the standard orthonormal basis in ℓ^2). The shift operator S satisfies the LRG condition with constant 1, but it is clearly not similar to a normal operator.

This example shows, that it is reasonable to apply the Resolvent Test only to operators with "thin" spectrum, where we can hope to "probe" spectral properties of the operator using only simplest rational functions as test functions (instead of all continuous functions, as in J. Wermer's theorem).

In [1], [6], [5] several results in this direction were obtained. For example, it was proved in [1] that if a perturbation $T = U + K$ where U is unitary and rank $K < \infty$ is *contractive* (i.e. $\|T\| \leq 1$), then T is similar to a normal operator if and only if the Linear Resolvent Growth condition (LRG) holds and the spectrum $\sigma(T)$ does not cover the unit disc $\mathbb{D}$. [1]

In this paper we consider the same situation, but *without* the assumption $\|T\| \leq 1$. We show that in this case the situation is completely different. Namely, the following theorem holds.

Theorem 0.1. *Given a unitary operator U with non-trivial absolutely continuous part of the spectrum, there exists a rank one perturbation*

$$T = U - (\,\cdot\,, a)b = U - ba^*$$

such, that the spectrum $\sigma(T)$ belongs to the unit circle $\mathbb{T}$, the operator T satisfies the Linear resolvent Growth (LRG) condition, but T is not similar to a normal (unitary)[2] operator.

Remark 0.2. The above theorem also true for rank one perturbation of self-adjoint operators (with non-trivial absolutely continous part of spectrum). One can just follow the proof for unitary operators, or use the Cayley transform. We leave details to the reader.

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[1] A simple example shows that it is possible to have $\sigma(T) = \text{clos}\,\mathbb{D}$ for a finite rak (rank 1) perturbation of a unitary operator. Let U be the *bilateral shift* i. e. the multiplication by the independent variable z in $L^2(\mathbb{T}) = L^2(\mathbb{T}, dm)$, and let $e_n(z) = z^n$, $n \in \mathbb{Z}$ be the standard orthonormal basis in $L^2(T)$. Then the operator $T = U - e_0 e_{-1}^* = U - (\,\cdot\,, e_{-1})e_0$ is unitarily equivalent to the orthogonal sum of the forward shift S and its adjoint S^*. So, $\sigma(T) = \text{clos}\,\mathbb{D}$ and T is clearly not similar to a normal operator.

[2] Any normal operator with spectrum on the unit circle is unitary.

The above theorem is the main result of the paper. The cases of unitary operators U with singular spectrum (continuous or discrete) represents an open problem, as well as the general question to describe all rank one perturbations of a unitary operator which are similar to a normal one. While there are several criteria of similarity to a *unitary* operator, see, for example, [8], in our situation (if one considers it in the spectral representation of the unitary operator U) they give rise to two weight estimates for the Hilbert Transform, and that is still an open problem. The general case, when eigenvalues appear in the complement of the unit circle is completely understood only when the resulting operator is a contraction: for general operators we do not know any results in this direction.

To complete the introduction let us recall that operators $T : H \to H$ having unitary spectrum ($\sigma(T) \subset \mathbb{T}$) and satisfying (LRG) but not similar to a unitary operator are known for a long time (see, for example, [7]), but in all these examples the "non-unitarity defects" $I - T^*T$, $I - TT^*$ are not controllable.

The paper is organized as follows: in Section 1 the problem is reduced to two weight estimates for the Hilbert Transform. Section 2 discusses F. Nazarov's counterexample to Sarason's conjecture, which is the basis of our counterexample. In section 3 we present the main construction, proving the theorem, stated above.

We tried to make the paper as self-contained as possible: in particular, we do not assume any familiarity with the perturbation theory. Although one can reduce the problem to the two weight estimate for the Hilbert Transform using, for example, S. Naboko's [8] criterion of similarity to a unitary operator, we do not use it: we use only elementary analytic methods.

1. PERTURBATIONS OF UNITARY OPERATORS

Let T be a perturbation of a unitary operator U, $T = U + K$. Then

$$(1.1) \qquad \lambda I - T = \lambda I - U - K = (\lambda I - U)(I - (\lambda I - U)^{-1}K)$$
$$= (\lambda I - U)(I - R_\lambda^U K)$$

We are interested in the case when K is a rank one operator,

$$K = (\,\cdot\,, a)b = ba^*.$$

In this case

$$I - R_\lambda^U K = I - (\,\cdot\,, a)R_\lambda^U b = I - (R_\lambda^U b)a^*.$$

Since the operator $I - ba^*$ is identity on $(\mathcal{L}\{a,b\})^\perp$, the determinant $\det(I - ba^*) = \det\big(I - (\,\cdot\,, a)b\big)$ is well defined.

Lemma 1.1.

$$\det(I - ba^*) = \det\big(I - (\,\cdot\,, a)b\big) = 1 - (b, a).$$

Proof. Pick an orthonormal basis in $\mathcal{L}\{a, b\}$ such that b has coordinates $(\beta, 0)$ in this basis. Let a has coordinates (α_1, α_2). Then in this basis

$$ba^* \,\Big|\, \mathcal{L}\{a, b\} = \begin{pmatrix} \beta \\ 0 \end{pmatrix} \begin{pmatrix} \overline{\alpha}_1 & \overline{\alpha}_2 \end{pmatrix} = \begin{pmatrix} \beta\overline{\alpha}_1 & \beta\overline{\alpha}_2 \\ 0 & 0 \end{pmatrix},$$

and therefore

$$\det(I - ba^*) = \det \begin{pmatrix} 1 - \beta\overline{\alpha}_1 & -\beta\overline{\alpha}_2 \\ 0 & 1 \end{pmatrix} = 1 - \beta\overline{\alpha}_1 = 1 - (b, a).$$

$\square$

Lemma 1.2. *Let* $K := ba^*$ *be a rank one operator. Operator* $I - K$ *is invertible if and only if* $d := \det(I - K) = 1 - (b, a) \neq 0$. *In this case*

$$(I - K)^{-1} = I + \frac{1}{d}K.$$

Proof. Note that $a^*b = (b, a)$, and therefore $K^2 = ba^*ba^* = (b, a)ba^* = (b, a)K$, and similarly $K^n = (b, a)^{n-1}K$. Therefore for $|\lambda| < 1/\|K\|$

$$(I - \lambda K)^{-1} = I + \lambda K + \lambda^2 K^2 + \dots$$

$$= I + \lambda K \cdot (1 + \lambda(b, a) + \lambda^2(b, a)^2 + \dots) = I + \frac{\lambda}{1 - \lambda(b, a)}K.$$

Taking analytic continuation to the point $\lambda = 1$ we get

$$(I - K)^{-1} = I + \frac{1}{1 - (b, a)}K = I + \frac{1}{d}K.$$

$\square$

Proposition 1.3. *Let* $T = U + K$, *where* $K = ba^*$. *For* $\lambda \notin \sigma(U)$ *let* $b_\lambda := R_\lambda^U b := (\lambda I - U)^{-1}b$, *and* $a_\lambda := (R_\lambda^U)^*a$, *and* $d(\lambda) = \det(I - b_\lambda a^*) = 1 - (b_\lambda, a)$.

Then, if $\lambda \notin \sigma(U)$ *the operator* $\lambda I - T$ *is invertible if and only if* $d(\lambda) \neq 0$, *and, moreover*

$$R_\lambda^T = (\lambda I - T)^{-1} = R_\lambda^U + \frac{1}{d(\lambda)}R_\lambda^U K R_\lambda^U = R_\lambda^U + \frac{1}{d(\lambda)} b_\lambda a_\lambda^*.$$

Proof. Formula (1.1) implies that

$$R_\lambda^T := (\lambda I - T)^{-1} = (I - R_\lambda^U K)^{-1} R_\lambda^U$$

and it follows from Lemma 1.2 that

$$(I - R_\lambda^U K)^{-1} = (I - (R_\lambda^U b)a^*)^{-1} = I + \frac{1}{d(\lambda)}R_\lambda^U K.$$

$\square$

To prove the main theorem (Theorem 0.1) it is sufficient to consider the case when U is a unitary operator with simple spectrum, so it can be realized as the multiplication by the independent variable z, on $L^2(\mu)$, where μ is a measure on the unit circle $\mathbb{T}$ (the spectral measure of T).

Indeed, any unitary operator (in a separable Hilbert space) can be represented as an orthogonal sum of operators with simple spectrum, and to prove the theorem it is sufficient to consider just one operator in the sum.

The assumption that the operator has non-trivial absolutely continuous part of the spectrum means that for at least one operator in the sum, the corresponding measure has non-trivial absolutely continuous part.

In fact, we can without loss of generality assume that the measure μ is absolutely continuous (consider only perturbations ba^* with with $a = b = 0$ a. e. with respect to the singular part μ_s of the measure μ). But for now we just assume that the operator U doesn't have isolated eigenvalues, i. e. that the measure μ doesn't have isolated atoms. In this case

$$\sigma(U) = \sigma_{\mathrm{ess}}(U) = \operatorname{supp}\mu,$$

where $\operatorname{supp}\mu$ stands for the *closed* support of the measure μ, and σ_{ess} is the essential specrum. Therefore

$$\sigma(U) = \sigma_{\mathrm{ess}}(U) = \sigma_{\mathrm{ess}}(T) \subset \sigma(T)$$

Since a unitary operator satisfies the linear resolvent growth condition (with constant 1), we have

$$\|R_\lambda^U\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(U))} \leq \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}.$$

So, to estimate the resolvent R_λ^T we need to estimate

$$\|R_\lambda^T - R_\lambda^U\| = \left\|\frac{1}{d(\lambda)} b_\lambda a_\lambda^*\right\| = \frac{1}{|d(\lambda)|}\|a_\lambda\| \cdot \|b_\lambda\|.$$

It is easy to compute that

$$b_\lambda = \left(R_\lambda^U b\right)(z) = \frac{b(z)}{\lambda - z}, \qquad z \in \mathbb{T}$$

and hence

$$\|b_\lambda\|^2 = \int_{\mathbb{T}} \frac{|f(z)|^2}{|\lambda - z|^2} d\mu(z) = \int_{\mathbb{T}} \frac{|f(z)|^2}{|1 - \lambda\overline{z}|^2} d\mu(z) = \left|1 - |\lambda|^2\right|^{-1} \mathcal{P}_\lambda(|b|^2 d\mu),$$

where $\mathcal{P}_\lambda(d\nu)$ denotes the Poisson (harmonic) extension of the measure ν at the point $\lambda \in \mathbb{C} \setminus \mathbb{T}$,

$$\mathcal{P}_\lambda(d\nu) := \int_{\mathbb{T}} \frac{|1 - |\lambda|^2|}{|1 - \lambda\overline{z}|^2} d\nu(z).$$

Similarly,

$$a_\lambda := \left((R_\lambda^U)^* a\right)(z) = a(z)/(\overline{\lambda - z}), \qquad z \in \mathbb{T},$$

and so

$$\|a_\lambda\|^2 = \left|1 - |\lambda|^2\right|^{-1} \mathcal{P}_\lambda(|a|^2 d\mu).$$

Notice, that the determinant $d(\lambda)$ also can be easily computed:

(1.2) $$d(\lambda) = 1 - (b_\lambda, a) = 1 - \int_{\mathbb{T}} \frac{b(z)\overline{a(z)}}{\lambda - z} d\mu(z).$$

Gathering everything together we get that the linear resolvent growth condition for T is equivalent to the estimate

$$(1.3) \quad \mathcal{P}_\lambda(|a|^2 d\mu) \cdot \mathcal{P}_\lambda(|b|^2 d\mu) \le C|d(\lambda)|^2 \frac{(1 - |\lambda|^2)^2}{\operatorname{dist}(\lambda, \sigma(T))^2}, \qquad \lambda \in \mathbb{C} \setminus \sigma(T)$$

In the case of unitary spectrum ($\sigma(T) \subset \mathbb{T}$), the last estimate can be further simplified using the following lemma by Y. Domar [3].

Lemma 1.4. *Let σ be a closed subset of the unit circle $\mathbb{T}$, and φ be a non-negative subharmonic function in $\mathbb{C} \setminus \sigma$ satisfying*

$$|\varphi(\lambda) \le \frac{1}{\operatorname{dist}(\lambda, \mathbb{T})} = \frac{1}{|1 - |\lambda||}, \qquad |\lambda| \ne 1.$$

Then

$$\varphi(\lambda) \le \frac{C}{\operatorname{dist}(\lambda, \sigma)}, \qquad \lambda \in \mathbb{C} \setminus \sigma,$$

where C is some absolute constant.

This lemma, applied to the subharmonic functions $\varphi = \varphi_{x,y}$, $\varphi_{x,y}(\lambda) = |(R_\lambda^T x, y)|$ implies that in the case of unitary spectrum the LRG condition $\|R_\lambda^T\| \le C / \operatorname{dist}(\lambda, \sigma(T))$ follows from (and so is equivalent to) a weaker one $\|R_\lambda^T\| \le C'/|1 - |\lambda||$. So, in the case of unitary spectrum the condition (1.3), which is equivalent to (LRG), can be replaced by

$$\mathcal{P}_\lambda(|a|^2 d\mu) \cdot \mathcal{P}_\lambda(|b|^2 d\mu) \le C|d(\lambda)|^2 (1 + |\lambda|)^2, \qquad \lambda \in \mathbb{C} \setminus \mathbb{T}.$$

Moreover, since for a measure ν on the circle $\mathbb{T}$ we have $\mathcal{P}_\lambda(d\nu) = \mathcal{P}_{1/\overline{\lambda}}(d\nu)$, the latter condition is equivalent to

$$(1.4) \qquad \mathcal{P}_\lambda(|a|^2 d\mu) \cdot \mathcal{P}_\lambda(|b|^2 d\mu) \le C|d(\lambda)|^2, \qquad \lambda \in \mathbb{D}.$$

To prove the theorem let us now make some simplifying assumptions.

As we already discussed above, we can assume without loss of generality that our unitary operator U is a multiplication by the independent variable z in $L^2(\mu)$ where μ is an absolutely continuous (with respect to Lebesgue measure) measure on the unit circle $\mathbb{T}$.

Moreover, we can always assume (again without losing generality) that $d\mu = \chi_E dm$, where E is a measurable set, and m is the normalized ($m(\mathbb{T}) = 1$) Lebesgue measure on $\mathbb{T}$. Indeed, if $d\mu = w \, dm$, we put $E := \{x \in \mathbb{T} : w(x) \ne 0\}$, and it is an easy exercise to show that the operators of multiplication by z in $L^2(\mu)$ and $L^2(\chi_E dm)$ are unitarily equivalent.

We will be considering only perturbations with the functions a and b having disjoint supports, i. e. satisfying $ab \equiv 0$,

In this case $d \equiv 1$, $\sigma(T) = \sigma(U) = \operatorname{Clos}(E) \subset \mathbb{T}$, so, the estimate (1.4) (and so (LRG)) can be rewritten as

$$\mathcal{P}_\lambda(|a|^2 dm) \cdot \mathcal{P}_\lambda(|b|^2 dm) \le C < \infty, \qquad \lambda \in \mathbb{D}.$$

Here we assume that a and b are supported on E, so we de not have to write $\chi_E dm$. In what follows we will even skip dm and write it as $\mathcal{P}(|a|^2)\mathcal{P}(|b|^2) \le C$.

Let us now find what does it mean in this case that T is similar to a normal operator. First of all, since $\sigma(T) \subset \mathbb{T}$, the normal operator has to be a unitary one.

So, let us $T = V\mathcal{U}V^{-1}$, where $\mathcal{U}$ is a unitary operator. By the Spectral Theorem (see [2]), $\mathcal{U}$ can be realized as the multiplication by the independent variable z in a Von Neuman integral

$$\mathbf{H} := \int_{\mathbb{T}} \oplus E(z) d\nu(z),$$

where ν is a scalar spectral measure of the operator $\mathcal{U}$. Fix such a realization.

Pick $f, g \in L^2(\chi_E dm)$, $\|f\| = \|g\| = 1$ (again we can assume that $f = g = 0$ outside of E). Let $\tilde{f} := V^{-1}f$, $\tilde{g} := V^*g$, $\tilde{f}, \tilde{g} \in \mathbf{H}$. Then

$$((\lambda I - T)^{-1}f, g) = ((\lambda I - \mathcal{U})^{-1}\tilde{f}, \tilde{g}) = \int_{\mathbb{T}} \frac{1}{\lambda - z} \cdot (\tilde{f}(z), \tilde{g}(z))_{E(z)} d\nu(z).$$

If we define the (complex-valued) measure $\tilde{\nu} = \tilde{\nu}_{f,g}$ by

$$\tilde{\nu}_{f,g}(X) = \int_X (\tilde{f}(z), \tilde{g}(z))_{E(z)} d\nu(z),$$

for Borel measurable subsets $X \subset \mathbb{T}$, then

$$(R^T_\lambda f, g) = \int_{\mathbb{T}} \frac{1}{\lambda - z} d\tilde{\nu}_{f,g}(z) = -\mathcal{C}[\nu_{f,g}](\lambda) = \mathcal{C}[-\nu_{f,g}](\lambda),$$

where $\mathcal{C}[\mu]$ stands for the Cauchy transform of a a measure μ on $\mathbb{T}$, $\mathcal{C}[\mu](\lambda) = \int_{\mathbb{T}}(z - \lambda)^{-1}d\mu$.

Note that

$$\mathrm{var}(\tilde{\nu}_{f,g}) = \int_{\mathbb{T}} |(\tilde{f}(z), \tilde{g}(z))_{E(z)}| d\nu(z)$$

$$\leq \int_{\mathbb{T}} \|\tilde{f}(z)\| \cdot \|\tilde{g}(z)\| d\nu(z)$$

$$\leq \left(\int_{\mathbb{T}} \|\tilde{f}(z)\|^2 d\nu(z)\right)^{1/2} \left(\int_{\mathbb{T}} \|\tilde{g}(z)\|^2 d\nu(z)\right)^{1/2}$$

$$= \|\tilde{f}\| \cdot \|\tilde{g}\| \leq \|V\| \cdot \|V^{-1}\|,$$

so for any f and g the *weak resolvent* $(R^T_\lambda f, g) = ((\lambda I - T)^{-1}f, g)$ is the Cauchy integral of a complex measure of bounded variation.

But such Cauchy transforms admit very simple description. Since

$$\mathcal{C}[\mu](\lambda) = \begin{cases} \displaystyle\sum_{n \geq 0} \widehat{\mu}(n+1)\lambda^n, & |\lambda| < 1 \\ \displaystyle-\sum_{n < 0} \widehat{\mu}(n+1)\lambda^n, & |\lambda| > 1 \end{cases}$$

where $\widehat{\mu}(k)$ denotes the Fourier coefficient of the measure μ, $\widehat{\mu}(k) := \int_{\mathbb{T}} \overline{z}^k d\mu(z)$, we have

$$\mathcal{C}[\mu](\lambda) - \mathcal{C}[\mu](1/\overline{\lambda}) = \mathcal{P}_\lambda(\overline{z}d\mu(z)).$$

Therefore an analytic function φ in $\mathbb{C} \setminus \mathbb{T}$ is a Cauchy transform of a (complex) measure μ (of bounded variation) if and only if its *harmonization* $\varphi^H(\lambda) := \varphi(\lambda) - \varphi(1/\overline{\lambda})$ is the Poisson extension of the measure $\overline{z}d\mu(z)$. And that happens if and only if $\sup_{0<r<1} \int_{\mathbb{T}} |\varphi(rz)|dm(z) < \infty$.

According to a Proposition 1.3, $R_\lambda^T = R_\lambda^U + b_\lambda a_\lambda^*$. We already know that for a unitary operator U the weak resolvent $(R_\lambda^U f, g)$ is a Cauchy transform of a measure, so we only have to check $(b_\lambda a_\lambda^* f, g) = (f, a_\lambda) \cdot (b_\lambda, g)$.

A simple computation gives us

$$(f, a_\lambda) = \int_E \frac{f(z)\overline{a}(z)}{\lambda - z} dm(z) = -\mathcal{C}[f\overline{a}dm](\lambda),$$

and similarly

$$(b_\lambda, g) = -\mathcal{C}[b\overline{g}dm](\lambda)$$

(all the functions a, b, f, g are supported on E).

Recall, that for $f \in L^2 = L^2(\mathbb{T}, dm)$ and $\lambda \in \mathbb{D}$

$$\mathcal{C}[fdm](\lambda) = P_+[\overline{z}f](\lambda), \qquad \mathcal{C}[fdm](1/\overline{\lambda}) = P_-[\overline{z}f](\lambda),$$

where P_+ and P_- denote the Riesz projections onto H^2 and H_-^2 respectively: $P_+f = \sum_{k\geq 0} \hat{f}(k)z^k$, $P_- = I - P_+$. Of course, we extend the functions P_+f, P_-f harmonically to $\mathbb{D}$.

Summarizing everything we get that the weak resolvent $(R_\lambda^T f, g)$ is the Cauchy transform of a complex measure (of bounded variation) if and only if the harmonic[3] in the unit disk $\mathbb{D}$ function

$$P_+(\overline{z}f\overline{a}) \cdot P_+(\overline{z}b\overline{g}) - P_-(\overline{z}f\overline{a}) \cdot P_-(\overline{z}b\overline{g})$$

is the Poisson extension of some complex measure of bounded variation on $\mathbb{T}$.

So, to show that Linear Resolvent Growth doesn't imply similarity to a normal operator it is enough to find 4 functions $a, b, f, g \in L^2$, all supported by the set E satisfying $ab \equiv 0$ and

$$\mathcal{P}_\lambda(|a|^2) \cdot \mathcal{P}_\lambda(|b|^2) \leq C,$$

and such, that the harmonic function $P_+(af) \cdot P_+(bg) - P_-(af) \cdot P_-(bg)$ is not a harmonic extension of a (complex) measure of bounded variation. To avoid unnecessary notation we have replaced here $\overline{z}g$ by g and $\overline{z}a$ by a.

To construct such functions we will use Nazarov's counterexample to a Sarason's conjecture about two weight estimates for Hilbert Transform.

2. NAZAROV'S COUNTEREXAMPLE TO SARASON'S CONJECTURE.

In [9] F. Nazarov proved the following result. Let $\mathcal{H}$ denote the Hilbert Transform

$$\mathcal{H}f(y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x)}{y - x} dx.$$

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[3]The first term in the function is a product of two analytic ones, and the second is the product of two antianalytic, so the function is indeed harmonic

Let a, b be functions on $\mathbb{R}$. We are interested when the operator $M_a \mathcal{H} M_b$, where M_a, M_b are multiplication operators by the functions a and b respectively, is a bounded operator on L^2. It was conjectured by D. Sarason, that the operator $M_a \mathcal{H} M_b$ is bounded if and only if

$$(2.1) \qquad \mathcal{P}_\lambda(|a|^2) \cdot \mathcal{P}_\lambda(|b|^2) \leq C < \infty \qquad \lambda \in \mathbb{C}_+;$$

here for a function φ defined on $\mathbb{R}$ the symbol $\mathcal{P}_\lambda(\varphi)$ stands for its harmonic extension at the point $\lambda \in \mathbb{C}_+$.

The above condition (2.1) is indeed necessary for the boundedness of the operator $M_a \mathcal{H} M_b$. But, as the following result by F. Nazarov asserts, this condition is not sufficient.

Theorem 2.1. *There exist functions a, b satisfying (2.1) and such, that the operator $M_a \mathcal{H} M_b$ is not bounded on L^2.*

Now let us make several simple observations. First of all

$$\|M_a \mathcal{H} M_b\| = \|M_{|a|} \mathcal{H} M_{|b|}\|,$$

i. e. the norm depends only on absolute values of a and b.

The second observation is that one can replace the Hilbert Transform $\mathcal{H}$ by the Riesz projection P_+, where P_+ is the orthogonal projection onto $H^2 = H^2(\mathbb{C}_+)$. Indeed, $P_+ = \frac{1}{2}(I + i\mathcal{H})$ and condition (2.1) implies that $ab \in L^\infty$, and therefore the operator $M_{ab} = M_a M_b$ is bounded.

Third observation is that it doesn't matter if we work on the real line or on the unit circle $\mathbb{T}$. Indeed, let us consider conformal mapping τ from $\mathbb{C}_+$ onto the unit disk $\mathbb{D}$, $\tau(z) = \frac{i-z}{i+z}$, and let $\mathcal{U} : L^2(\mathbb{T}) \to L^2(\mathbb{R})$ be a unitary mapping, $\mathcal{U} f := \frac{1}{\sqrt{\pi}} \frac{1}{x+i} f(\tau(x))$. Clearly (and it is a very well known fact), $\mathcal{U} H^2(\mathbb{D}) = H^2(\mathbb{C}_+)$, and

$$\mathcal{U}^{-1} M_a P_+^{\mathbb{C}_+} M_b \mathcal{U} = M_{a_1} P_+^{\mathbb{D}} M_{b_1},$$

where $a_1 := a \circ \tau^{-1}$, $b_1 := b \circ \tau^{-1}$. Since the condition (2.1) is conformally invariant, a, b satisfy it if and only if

$$\sup_{\lambda \in \mathbb{D}} \mathcal{P}_\lambda(|a_1|^2) \cdot \mathcal{P}_\lambda(|b_1|^2) < \infty.$$

And final observation is that given an interval (an arc) $\mathcal{I} \subset \mathbb{T}$ one can find functions a, b satisfying (2.1), supported by $\mathcal{I}$ and such, that the operator $M_a P_+ M_b$ is not bounded. This is easier to see on the circle $\mathbb{T}$. Take the functions a, b on the circle $\mathbb{T}$, and consider a smooth finite partition of unity $1 = \sum_1^n \varphi_k$, $0 \leq \varphi_k \leq 1$, such that each φ_k is supported by a closed interval of length at most $|\mathcal{I}|/2$. Then

$$M_a P_+ M_b = \sum_{1 \leq j,k \leq n} M_{a\varphi_k} P_+ M_{b\varphi_j},$$

so at least one of the operators $M_{a\varphi_k} P_+ M_{b\varphi_j}$ is unbounded. This can happen only if (closed) supports of φ_k and φ_j have nontrivial intersection. Therefore

both functions $\varphi_k a$ and $\varphi_j b$ are supported by one interval of length at most $|\mathcal{I}|$, and rotating the picture completes the proof.

So, we get the result we will be using.

Theorem 2.2. *Given an arc $\mathcal{I} \subset \mathbb{T}$ there exist functions $a, b \in L^2 = L^2(\mathbb{T})$ supported by $\mathcal{I}$, and satisfying*

$$(2.2) \qquad \sup_{\lambda \in \mathbb{D}} \mathcal{P}_\lambda(|a|^2) \cdot \mathcal{P}_\lambda(|b|^2) < \infty,$$

and such, that the operator $M_a P_+ M_b$ is not bounded on L^2.

Remark 2.3. The fact that $a, b \in L^2 = L^2(\mathbb{T})$ follows immediately from (2.2).

3. Constructing the counterexample

Consider a sequence of arcs $\mathcal{I}_n \subset \mathbb{T}$, $|\mathcal{I}_n| = 2^{-n-5}$ such that the arcs $10\mathcal{I}_n$ are disjoint. Here, given an interval (arc) $\mathcal{I}$ we denote by $k\mathcal{I}$ the interval (arc) with the same center and of length $k \cdot |\mathcal{I}|$.

Let μ be a complex measure (of bounded variation) on $\mathbb{T}$, and let $\Phi = \Phi_\mu$ be it harmonic extension into the unit disk $\mathbb{D}$. It is an easy exercise to check that for any sequence r_n, $1 - |\mathcal{I}_n| \le r_n < 1$

$$\sum_{n=1}^{\infty} \int_{2\mathcal{I}_n} |\Phi(r_n \theta)| dm(\theta) \le C \operatorname{var} \mu$$

with some constant C, not depending on $\mathcal{I}_n$, r_n, μ as long as they satisfy the above assumptions.

Now let us describe the main component of the construction. Let E be a subset of $\mathbb{T}$ of positive measure. Recall, that a point of $x \in E$ is called a (Lebesgue) density point if

$$\lim_{|\mathcal{I}| \to 0, \mathcal{I} \ni x} \frac{m(E \cap \mathcal{I})}{m(\mathcal{I})} = 1,$$

and that almost all points of E are density points.

Proposition 3.1. *Given a density point $x_0 \in E$ and numbers $R, \varepsilon, \delta > 0$, one can find an arbitrary small arc $\mathcal{I} \subset \mathbb{T}$ ($|\mathcal{I}| < \delta$) centered at x_0 and bounded functions a, b, f, g supported by $\mathcal{I} \cap E$, such that $ab \equiv 0$,*

$$(3.1) \qquad \sup_{\lambda \in \mathbb{C}_+} \mathcal{P}_\lambda(|a|^2)\mathcal{P}_\lambda(|b|^2) \le 1, \qquad \|f\|_2 \le \varepsilon, \qquad \|g\|_2 \le \varepsilon,$$

and such, that for some r, $1 - |\mathcal{I}| \le r < 1$ we have for the functions $F := P_+(af)P_+(bg)$, $G := P_-(af)P_-(bg)$

$$\int_{2\mathcal{I}} |F(r\theta)| dm(\theta) > R, \qquad \int_{2\mathcal{I}} |G(r\theta)| dm(\theta) < \varepsilon.$$

Proof. Let $\mathcal{I}$ be a small arc ($|\mathcal{I}| < \delta$) centered at x. By Theorem 2.2 there exist functions $a, b \in L^2$ supported by the interval $\mathcal{I}$ such that the operator $M_b P_+ M_a$ is unbounded. Then there exists a function $f \in L^2$ such that $bP_+(af) \notin L^2$. That means for some $g \in L^2$ the function $bgP_+(af)$ is not in L^1. So we can pick *bounded* (L^∞) functions a, b, f, g supported by the interval $\mathcal{I}$, satisfying (3.1) and such, that $\|bgP_+(af)\|_1 > 4R$. Replacing g by $z^n g(z)$, where $n > 0$ is sufficiently large number we can make the norm $\|P_-(bg)\|_2$ as small as we want, so the norms $\|P_-(bg)P_+(af)\|_1$, $\|P_-(bg)P_-(af)\|_1$ can be as small as we want, and the norm $\|P_+(bg)P_+(af)\|_1$ can be as close to the norm $\|bgP_+(af)\|_1$ as we want. So we can always achieve

$$\|P_+(af)P_+(bg)\|_1 > 3R, \qquad \|P_-(af)P_-(bg)\|_1 < \varepsilon/2.$$

Now we want all the functions to be supported by $\mathcal{I} \cap E$. Since all functions are bounded, so we can assume that

$$\|a\|_\infty, \|b\|_\infty \le C, \qquad \|f\|_\infty, \|g\|_\infty \le C/m(\mathcal{I})^{1/2}.$$

Then there exists a small constant $\alpha > 0$ (depending on C) such that if $m(\mathcal{I} \setminus E) < \alpha m(\mathcal{I})$, then

$$\|P_+(\chi_E af)P_+(\chi_E bg)\|_1 > 2R, \qquad \|P_-(\chi_E af)P_-(\chi_E bg)\|_1 < \varepsilon$$

(the functions $\chi_E af$, $\chi_E bg$ are close in L^2-norm to af and bg respectively). So, if $m(\mathcal{I} \setminus E) < \alpha m(\mathcal{I})$ we can just multiply all the functions by χ_E.

It looks like a bad circle here, since we first picked the interval $\mathcal{I}$, then the functions a, b, f, g (and so the constant C), and now we want $m(\mathcal{I} \setminus E)/m(\mathcal{I}) < \alpha = \alpha(C)$. However, we can break this bad circle, because everything can be rescaled.

This can be better explained on the real line $\mathbb{R}$. Recall, that, as we already discussed above in Section 2, we can use conformal mapping from the disk to the upper half-plane to move freely from the circle to the real line and back. So let everything is on the real line. Without loss of generality we can assume that our density point x_0 is the origin 0.

Since the Riesz Projection P_+ commutes with the scaling operators $f(z) \mapsto f(kz)$ (for all $k \in \mathbb{R} \setminus \{0\}$), nothing changes when we rescale the picture, i. e. replace $a(x)$, $b(x)$ by $a(kx)$, $b(kx)$ and $f(x)$, $g(x)$ by $k^{1/2}f(kx)$, $k^{1/2}g(kx)$ respectively. Nothing, except the fact that the interval $\mathcal{I}$ will be replaced by $k^{-1}\mathcal{I}$, so now we can make $m(\mathcal{I} \setminus E)/m(\mathcal{I})$ as small as we want, because 0 is a density point of E.

Now let us pick radius $r \ge 1 - |\mathcal{I}|$ sufficiently close to 1 such that for functions $F := P_+(af)P_+(bgb)$, $G := P_-(ag)P_-(fb)$ we have

$$\int_{2\mathcal{I}} |F(r\theta)| dm(\theta) > 3R, \qquad \int_{2\mathcal{I}} |G(r\theta)| dm(\theta) < \varepsilon/2.$$

We are almost done, the only problem is that $ab \not\equiv 0$.

To correct that, define

$$\chi(e^{it}) = \left\{ \begin{array}{ll} 1, & 0 \le t < \pi \\ 0 & \pi \le t < 2\pi \end{array} \right.$$

and let $\chi_n(z) := \chi(z^n)$. Our claim is, that if we replace a by $\chi_n a$ and b by $(1 - \chi_n)b$ with sufficiently large n, then we are done.

Indeed, the functions χ_n, $(1 - \chi_n)$ converge weakly in L^2 to the function $\frac{1}{2}$ as $n \to \infty$ (the sequence χ_n is bounded, and $\int f\chi_n dm \to \frac{1}{2}\int f dm$ for smooth f). Since for a function $\varphi \in L^2$,

$$(3.2) \qquad P_+\varphi(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\varphi(\xi)}{\xi - z} d\xi,$$

one can conclude that the functions $F_n := P_+(\chi_n ag)P_+((1-\chi_n)bg)$ converge to $\frac{1}{2}F$ uniformly on compact subsets of $\mathbb{D}$.

Similarly, the functions $G_n := P_-(\chi_n af)P_-((1-\chi_n)bg)$ converge to $\frac{1}{2}G(z)$ uniformly on compact subsets of $\mathbb{D}$. So, for sufficiently large n,

$$\int_{2\mathcal{I}} |F_n(r\theta)| dm(\theta) > R, \qquad \int_{2\mathcal{I}} |G_n(r\theta)| dm(\theta) < \varepsilon.$$

Replace a by $\chi_n a$ and b by $(1 - \chi_n)b$ with sufficiently large n, then we are done! $\qquad \square$

Using Proposition 3.1 take the sequence of intervals $\mathcal{I}_n \subset \mathbb{T}$ as above, and construct bounded functions a_n, b_n, f_n, g_n supported by $\mathcal{I}_n \cap E$ and radii r_n, $1 - |\mathcal{I}_n| \le r_n < 1$ such, that

1. $a_n b_n \equiv 0$;
2. $\mathcal{P}_\lambda(|a_n|^2)\mathcal{P}_\lambda(|b_n|^2) \le 1$ for all $\lambda \in \mathbb{D}$;
3. $\|f_n\|_2 \le 2^{-n}$, $\|g_n\|_2 \le 2^{-n}$;

and such that for the functions $F_n := (P_+(a_n f_n)P_+(b_n g_n))$, $G_n := (P_-(a_n f_n)P_-(b_n g_n))$ we have

4. $\displaystyle\int_{2\mathcal{I}_n} |F_n(r_n\theta)| dm(\theta) > 2^n, \qquad \int_{2\mathcal{I}_n} |G_n(r_n\theta)| dm(\theta) < 2^{-n}.$

The condition 2 implies that

$$\frac{1}{|\mathcal{I}_n|} \int_{\mathcal{I}_n} |a_n|^2 dm \cdot \frac{1}{|\mathcal{I}_n|} \int_{\mathcal{I}_n} |b_n|^2 dm \le C$$

where $C < \infty$ is some absolute constant, so we can always assume that

5. $\displaystyle\frac{1}{|\mathcal{I}_n|} \int_{\mathcal{I}_n} |a_n|^2 dm \le C, \qquad \frac{1}{|\mathcal{I}_n|} \int_{\mathcal{I}_n} |b_n|^2 dm \le C.$

Define

$$a = \sum a_n \qquad b := \sum b_n, \qquad f := \sum f_n, \qquad g = \sum g_n.$$

It is easy to check that the pair a, b satisfies the condition (2.1). Indeed, if $\text{dist}(\lambda, \mathcal{I}_k) \ge |\mathcal{I}_k|$ for all k, then, computing $\mathcal{P}_\lambda(|a|^2)$, we can replace each

$|a_k|^2$ on the interval $\mathcal{I}_k$ by its average, and since all the averages are bounded (condition 5), we get $\mathcal{P}_\lambda(|a|^2) \le C < \infty$, and the same for $\mathcal{P}_\lambda(|b|^2)$.

If $\mathrm{dist}(\lambda, \mathcal{I}_k) < |\mathcal{I}_k|$ for some k, then $\mathrm{dist}(\lambda, \mathcal{I}_j) \ge |\mathcal{I}_k|$ for $j \ne k$. We can estimate

$$\mathcal{P}_\lambda(|a|^2) = \mathcal{P}_\lambda(|a_k|^2) + \mathcal{P}_\lambda(\sum_{j \ne k} |a_j|^2) \le \mathcal{P}_\lambda(|a_k|^2) + C \cdot (1 - |\lambda|)/|\mathcal{I}_k|,$$

and similarly for $\mathcal{P}_\lambda(|b_k|^2)$,

$$\mathcal{P}_\lambda(|b|^2) \le \mathcal{P}_\lambda(|b_k|^2) + C \cdot (1 - |\lambda|)/|\mathcal{I}_k|.$$

Recall, that the Poisson kernel P_λ is given by

$$P_\lambda(z) = \frac{1 - |\lambda|^2}{|1 - \overline{\lambda}z|^2},$$

so $(1 - |\lambda|) \cdot P_\lambda(z) \le 1$. Therefore

$$\mathcal{P}_\lambda(|a_k|^2) \cdot (1 - |\lambda|)/|\mathcal{I}_k| \le C \cdot |\mathcal{I}_k|^{-1} \int_{\mathcal{I}_k} |a_k|^2 dm \le C.$$

and similarly $\mathcal{P}_\lambda(|a_k|^2) \cdot (1 - |\lambda|)/|\mathcal{I}_k| \le C$. The other two terms in the product

$$\left(\mathcal{P}_\lambda(|a_k|^2) + C \cdot (1 - |\lambda|)/|\mathcal{I}_k|\right)\left(\mathcal{P}_\lambda(|b_k|^2) + C \cdot (1 - |\lambda|)/|\mathcal{I}_k|\right)$$

are also trivially bounded, so indeed $\mathcal{P}_\lambda(|a|^2)\mathcal{P}_\lambda(|b|^2) \le C < \infty$.

Let us check that the harmonic function

$$F - G = P_+(af)P_+(bg) - P_-(af)P_-(bg)$$

is not a harmonic extension of a complex measure μ of bounded variation. Formula (3.2) implies that for $z \in \mathbb{D}$, $\mathrm{dist}(z, \mathcal{I}_k) \ge |\mathcal{I}_k|$ we have the estimate

$$|P_+(a_k f_k)(z)| \le \frac{C}{|\mathcal{I}_k|}\|a_k f_k\|_1 \le \frac{C}{|\mathcal{I}_k|}\|a_k\|_2 \cdot \|f_k\|_2$$

$$\le \frac{C}{|\mathcal{I}_k|}|\mathcal{I}_k|^{1/2}2^{-k} \le C2^{-k/2}.$$

and similar estimate holds for $|P_+(b_k g_k)|$. Therefore, for $\theta \in \mathcal{I}_n$,

$$\sum_{k \ne n}\left|\left(P_+(af)P_+(bg)\right)(r_n\theta)\right| \le C\sum_{k=1}^\infty 2^{-k} \le C,$$

and similarly

$$\sum_{k \ne n}\left|\left(P_-(af)P_-(bg)\right)(r_n\theta)\right| \le C.$$

Therefore for the functions $F_k := (P_+(a_k f_k)P_+(b_k g_k)$, $G_k := (P_-(a_k f_k)P_-(b_k g_k)$

$$\sum_{k \ne n}\int_{2\mathcal{I}_n}|F_k(r_n\theta)|dm(\theta) \ge 2^n - C2^{-n}$$

and

$$\sum_{k \neq n} \int_{2\mathcal{I}_n} |G_k(r_n\theta)|dm(\theta) \leq 2^{-n} + C2^{-n}$$

Therefore for $F = P_+(af)P_+(bg) = \sum_n F_n$ and $G_n = P_-(af)P_-(bg) = \sum_n G_n$, we have

$$\sum_n \int_{2\mathcal{I}_n} |F(r_n\theta) - G(r_n\theta)|dm(\theta) = +\infty$$

Therefore, $F - G$ cannot be the harmonic extension of a (complex) measure of bounded variation, so we got a counterexample. $\square$

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