

SCALAR AND VECTOR MUCKENHOUPPT WEIGHTS

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ABSTRACT. We inspect the relationship between the $\mathcal{A}_{p,q}$ condition for families of norms on vector valued functions and the A_p condition for scalar weights. In particular we will show if we are considering a norm-valued function $\rho_{(\cdot)}$ such that, uniformly in all nonzero vectors x , $\rho_{(\cdot)}(x)^p \in A_p$ and $\rho_{(\cdot)}^*(x)^q \in A_q$ then the following hold: If $p = q = 2$, and functions take values in \mathbb{R}^2 then $\rho \in \mathcal{A}_{2,2}$. If $p = q = 2$ and functions take values in \mathbb{R}^d , $d \geq 6$, ρ need not be an $A_{2,2}$ weight. If ρ satisfies the relatively weak $A_{0,0}$ condition in addition to the scalar conditions mentioned above, then $\rho \in \mathcal{A}_{p,q}$.

0. INTRODUCTION

The famous Hunt–Muckenhoupt–Wheeden (Coifman–Fefferman) Theorem states that the Hilbert Transform T is a bounded operator in the weighted space $L^p(w)$ ($1 < p < \infty$) if and only if the weight satisfies the so-called Muckenhoupt A_p condition:

$$(A_p) \quad \sup_{I \subset \mathbb{R}} \left(\int_I w \right) \left(\int_I w^{-q/p} \right)^{p/q} < \infty;$$

Here the supremum is taken over all intervals I of the real line (of \mathbb{R}^n), $1/p + 1/q = 1$, and $\int_I f(t) dt = |I|^{-1} \int_I f(t) dt$. The above supremum is called *the Muckenhoupt A_p norm* of the weight w , and will be denoted in what follows by $[w]_{A_p}$.

While this is standard notation, we nevertheless recall that the norm in the weighted space $L^p(w)$ is defined by

$$\|f\|_{L^p(w)}^p = \int |f(t)|^p w(t) dt \quad (1 \leq p < \infty).$$

One can easily generalize the notion of a weighted L^p space to the case of vector-valued functions (with values in \mathbb{C}^d or \mathbb{R}^d , $d < \infty$) and matrix-valued weights. Namely, if W is a matrix-valued weight, i.e. a function (on \mathbb{R} or \mathbb{R}^n) whose values are positive semidefinite matrices, then it is natural to define the norm in the weighted space $L^p(W)$ as

$$\|f\|_{L^p(W)}^p = \int \|W^{1/p}(t)f(t)\|^p dt.$$

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For $p = 2$ this norm looks especially nice,

$$\|f\|_{L^2(W)}^2 = \int (W(t)f(t), f(t))dt.$$

For other p this definition also looks quite natural, especially if one notices that (under the assumption that the matrices $W(t)$ are invertible a.e.) the boundedness of an operator T in the weighted space $L^p(W)$ is equivalent to the boundedness of the product $M_W^{1/p} T M_W^{-1/p}$ in the non-weighted L^p : here M_W stands for the multiplication operator, $M_W f := Wf$.

Motivated by problems in the theory of stationary processes, in [7] Treil and Volberg introduced the matrix analogue \mathbf{A}_2 of the Muckenhoupt A_2 condition. Namely, they proved (under the assumption that the average of the weight over some interval is invertible) that the Hilbert Transform is bounded if and only if the weight W satisfies the condition

$$(\mathbf{A}_2) \quad \sup_I \left\| \left(\int_I W(t) dt \right)^{1/2} \left(\int_I W(t)^{-1} dt \right)^{1/2} \right\| \leq C < \infty$$

(here again the supremum is taken over all intervals $I \subset \mathbb{R}$). Then in [5] and [8] this condition was generalized to the case of other exponents p , and the condition $\mathcal{A}_{p,q}$ was introduced. To explain this condition one needs to introduce some notation.

First, for reasons which will soon be clear, it is more convenient to work in more general situation of norm-valued weights. Namely, let $t \mapsto \rho_t$, $t \in \mathbb{R}$ be a function whose values are norms (or even seminorms) on \mathbb{R}^d or \mathbb{C}^d . We assume this function to be measurable in the sense that for any vector $x \in \mathbb{R}^d$ the function $t \mapsto \rho_t(x)$ is measurable. For the sake of brevity we will use simply the symbol ρ for such functions. To avoid confusion we will use the bold symbol $\boldsymbol{\rho}$ in cases when we need a constant norm on \mathbb{R}^d or \mathbb{C}^d .

So, let ρ be a measurable norm-valued function. The norm in the weighted space $L^p(\rho)$ is defined by

$$\|f\|_{L^p(\rho)}^p := \int \rho_t(f(t))^p dt.$$

The weighted space $L^p(W)$ with a matrix weight W is a particular case of the space $L^p(\rho)$ with ρ_t defined by $\rho_t(x) = \|W(t)^{1/p}x\|$.

For a constant norm (seminorm) $\boldsymbol{\rho}$ on \mathbb{R}^d (or \mathbb{C}^d) let $\boldsymbol{\rho}^*$ denote the dual norm,

$$\boldsymbol{\rho}^*(x) = \sup_{y \neq 0} \frac{|(x, y)|}{\boldsymbol{\rho}(y)}.$$

Here (x, y) stands for the standard inner product in \mathbb{R}^d (or \mathbb{C}^d). Note, that if $\boldsymbol{\rho}$ is only a seminorm, $\boldsymbol{\rho}^*(x)$ can be infinite.

If ρ is a norm-valued function, we define the function ρ^* by $\rho_t^* = (\rho_t)^*$. For a norm-valued function ρ we denote by $\langle \rho \rangle_{I,p}$ its p -average,

$$\langle \rho \rangle_{I,p}(x) := \left(\int_I \rho_t(x)^p dt \right)^{1/p}, \quad x \in \mathbb{R}^d.$$

This expression makes sense for all $p \in (0, \infty)$, although only for $p \geq 1$ will the result be a convex function on \mathbb{R}^d , i.e. a (semi)norm.

We will also use the notation $\langle \rho \rangle_{I,0}$ for the geometric mean of ρ ,

$$\langle \rho \rangle_{I,0}(x) := \exp \left\{ \int_I \ln \rho_t(x) dt \right\}, \quad x \in \mathbb{R}^d.$$

Using the introduced notation, let us define the $\mathcal{A}_{p,q}$ condition as

$$(\mathcal{A}_{p,q}) \quad \langle \rho^* \rangle_{I,q} \leq C \langle \rho \rangle_{I,p}^*$$

(the q -average of the dual norm is dominated by the dual of p -average)¹ for all intervals (cubes) I . Note that if $1/p + 1/q = 1$ then the opposite inequality always holds with $C = 1$.

Nazarov–Treil [5] and Volberg [8] proved that the Hilbert Transform is bounded for in the weighted space $L^p(\rho)$ ($1 < p < \infty$) if and only if the weight ρ satisfies the above condition $\mathcal{A}_{p,q}$ with $1/p + 1/q = 1$.

Note that in the scalar case, if the norm ρ_t is given by $\rho_t(x) = w(t)^{1/p}|x|$, the $\mathcal{A}_{p,q}$ condition for ρ ($1/p + 1/q = 1$) is equivalent to the classical Muckenhoupt condition A_p for w . Namely, the condition $\mathcal{A}_{p,q}$ can be rewritten in this case as

$$\sup_{I \subset \mathbb{R}} \left(\int_I w \right)^{1/p} \left(\int_I w^{-q/p} \right)^{1/q} \leq C < \infty;$$

Also, for $p = 2$ and the norm-valued function ρ given by the matrix weight W , $\rho_t(x) = (W(t)x, x)^{1/2} = \|W(t)^{1/2}x\|$, the condition $\mathcal{A}_{2,2}$ for ρ is equivalent to the matrix Muckenhoupt condition \mathbf{A}_2 for W , see [5] for the detailed discussion.

The $\mathcal{A}_{p,q}$ condition looks rather complicated, and it is natural to ask whether it is possible to simplify it. In particular, whether it is possible to reduce it to the classical scalar A_p condition.

One direction is easy. As it was discussed in [5], for a fixed interval I the best constant C in the estimate $\langle \rho^* \rangle_{I,q} \leq C \langle \rho \rangle_{I,p}^*$ is exactly the norm of the averaging operator $f \mapsto \chi_I \cdot \int_I f$ in $L^p(\rho)$. So, if we restrict our attention to the functions of form $f = \varphi x$, where φ is a scalar function, and x is a constant vector, then one can conclude that the condition $\rho \in \mathcal{A}_{p,q}$, $1/p + 1/q = 1$ implies that the weight $w = w_x$, $w_x(t) = \rho_t(x)$ is a scalar Muckenhoupt A_p weight. Moreover, its Muckenhoupt norm $[w_x]_{A_p}$ can be estimated by C^p , where C is the constant from the $\mathcal{A}_{p,q}$ condition.

¹For the dual of average we will use the shorter notation $\langle \rho \rangle_{I,p}^*$ instead of more “grammatically correct” but longer $(\langle \rho \rangle_{I,p})^*$

Therefore, one can conclude that the scalar weights w_x , $x \in \mathbb{R}^d$ are *uniformly* A_p in the sense that their Muckenhoupt norms are uniformly bounded. (Note that this statement can be proven directly, see Lemma 1.5 below).

Since a norm $\rho \in \mathcal{A}_{p,q}$ if and only if $\rho^* \in \mathcal{A}_{q,p}$ (with the same constant), the same conclusion can be made about the weights v_x , $x \in \mathbb{R}^d$, $v_x(t) = \rho_t^*(x)^q$, namely that the weights v_x are uniformly A_q weights, $1/p + 1/q = 1$. The main question addressed in this paper is whether this implication can be reversed. That is, we want to determine under what conditions knowing the w_x and v_x are uniformly A_p and A_q respectively allows us to conclude that $\rho \in \mathcal{A}_{p,q}$. The remainder of the paper is dedicated to answering this question in several instances.

Section 1 addresses some further properties of dual norms, A_p scalar weights, and $\mathcal{A}_{p,q}$ norms that will be useful in our calculations. In section 2 we will discuss how an additional assumption of the $\mathcal{A}_{0,0}$ condition, the weakest of the $\mathcal{A}_{r,s}$ conditions, on ρ will allow us to bridge the gap between the scalar A_p and A_q conditions on w_x and v_x and the $\mathcal{A}_{p,q}$ condition on ρ . The gap can also be bridged if the range has real dimension 2 and $p = q = 2$, as will be shown in section 3. In the final two sections we will show that when the dimension of the range is at least 6 and $p = q = 2$ then the $\mathcal{A}_{p,q}$ condition on ρ is a stronger condition than the uniform A_p and A_q conditions on w_x and v_x . First we will construct an example with a domain of SO_6 and then use a space filling curve to convert this to an example with domain \mathbb{R}^1 .

1. SOME FACTS ABOUT DUAL NORMS, A_p , AND $\mathcal{A}_{p,q}$.

In this section we establish some basic properties of dual norms, A_p weights and $\mathcal{A}_{p,q}$ norms on vector spaces.

We begin with a few properties of dual norms. Let ρ be a norm on some vector space \mathcal{X} . Denote by \mathcal{Y} the dual space of \mathcal{X} . We recall from the introduction that the dual norm, ρ^* , to ρ is given by

$$(1.1) \quad \rho^*(y) = \sup_{x \neq 0} \frac{|(y, x)|}{\rho(x)}$$

where (y, x) denotes the dual pairing. For our purposes $\mathcal{X} = \mathcal{Y} = \mathbb{C}^d$ or \mathbb{R}^d and (y, x) is the usual inner product. Since \mathbb{C}^d and \mathbb{R}^d are reflexive, $\rho = \rho^{**}$ for any norm. In some sense, a dual norm is like the inverse of a positive matrix; it is easy to check that if $\rho(x) = \|Wx\|$ then $\rho^*(x) = \|W^{-1}x\|$. We also observe that taking dual norms switches the direction of inequalities.

Lemma 1.1. *If $\rho_1 \leq \rho_2$ in the sense that $\rho_1(x) \leq \rho_2(x)$ for all x then $\rho_2^* \geq \rho_1^*$ (in the same sense).*

Proof. Notice that in the definition of dual norms, computing ρ_2^* involves taking a supremum over a larger set than computing ρ_1^* . \square

1.1. **A partial ordering of $\mathcal{A}_{r,s}$ conditions.** From the introduction we recall the $\mathcal{A}_{p,q}$ condition:

$$(\mathcal{A}_{p,q}) \quad \langle \rho^* \rangle_{I,q} \leq C \langle \rho \rangle_{I,p}^*$$

Furthermore, we note that this condition still makes sense if p and q are not conjugate exponents. Moreover, it can be defined for all $p, q \in [0, 1)$, not only for $p, q \geq 1$ (recall that $\langle \rho \rangle_{I,0}$ denotes the geometric mean). We should also notice that although for $p < 1$ the average $\langle \rho \rangle_{I,p}$ is not necessarily a norm, it is still a 1-homogeneous function, so the dual norm is well defined and is still a norm.

Let us use the exponents r and s to emphasize that r and s are not necessarily conjugate exponents. If $r > \tilde{r} \geq 0$ then using Hölder's inequality (Jensen's inequality if $\tilde{r} = 0$) we get:

$$(1.2) \quad \langle \rho \rangle_{I,\tilde{r}}(x) = \left(\int_I (\rho_t(x))^{\tilde{r}} dt \right)^{1/\tilde{r}} \leq \left(\int_I (\rho_t(x))^r dt \right)^{\tilde{r}/(r\tilde{r})} = \langle \rho \rangle_{I,r}(x).$$

Taking dual norms allows us to also say that if $r > \tilde{r} \geq 0$ then $\langle \rho_t \rangle_{I,r}^*(x) \leq \langle \rho_t \rangle_{I,\tilde{r}}^*(x)$.

Putting these two inequalities together gives us the following lemma:

Lemma 1.2. *If $r > \tilde{r} \geq 0$, $s > \tilde{s} \geq 0$ and $\rho \in \mathcal{A}_{r,s}$ then $\rho \in \mathcal{A}_{\tilde{r},\tilde{s}}$.*

Proof.

$$\langle \rho^* \rangle_{I,\tilde{s}} \leq \langle \rho^* \rangle_{I,s} \leq C \langle \rho \rangle_{I,r}^* \leq C \langle \rho \rangle_{I,\tilde{r}}^*$$

□

In this sense $\mathcal{A}_{0,0}$ is the weakest $\mathcal{A}_{r,s}$ condition.

We continue with a discussion of how dual norms and averages of norms relate. The following useful lemma can be derived directly from the definition of a dual norm.

Lemma 1.3. *Let ρ be a norm on a Hilbert space \mathcal{E} and $x, y \in \mathcal{E}$. Then $|(x, y)| \leq \rho(x)\rho^*(y)$.*

In the following we will find it useful to rearrange Lemma 1.3 as follows:

$$(1.3) \quad \frac{1}{\rho(x)} \leq \frac{\rho^*(y)}{|(x, y)|}.$$

We will also need the following lemma.

Lemma 1.4. *If $1/p + 1/q = 1$ then*

$$\langle \rho^* \rangle_{I,q}(x) \geq \langle \rho \rangle_{I,p}^*(x)$$

Proof. Applying Hölder's inequality we get

$$(1.4) \quad |(x, y)| \leq \frac{1}{|I|} \int_I \rho_t(y)\rho_t^*(x)dt \leq \langle \rho(x) \rangle_{I,p} \langle \rho^*(y) \rangle_{I,q}$$

Now we divide by $\langle \rho^*(y) \rangle_{I,q}$ and take the supremum over all $y \neq 0$, yielding

$$\langle \rho^*(x) \rangle_{I,q}^* \leq \langle \rho(x) \rangle_{I,p}$$

Taking the dual norms of both sides reverses the inequality. \square

Note that the $\mathcal{A}_{p,q}$ condition tell us that a norm satisfies the reverse inequality (up to a constant).

1.2. A_∞ and $\mathcal{A}_{p,0}$ conditions. Let us recall the classical A_∞ condition for scalar weights. A weight w is said to satisfy A_∞ condition if it satisfies a reverse Jensen inequality

$$\int_I w(t) dt \leq C \exp \left\{ \int_I \ln w(t) dt \right\}.$$

Note, that very often a different definition of A_∞ is used; the above definition is equivalent to the classical one.

If we use our “norm-valued notation”, $\rho_t = w(t)^{1/p}$ (formally we should write $\rho_t(x) = w(t)^{1/p}|x|$, $x \in \mathbb{R}^1$), then the $w \in A_\infty$ if and only if ρ satisfies $\mathcal{A}_{p,0}$ condition discussed above.

Using the discussed above ordering of $\mathcal{A}_{r,s}$ conditions, one can conclude that if the weight $w \in \mathcal{A}_p$ implies $w \in A_\infty$. This fact is very well known in classical (scalar) harmonic analysis, and we just gave an alternative proof of it.

1.3. Why $\rho \in \mathcal{A}_{p,q}$ implies $(\rho(x))^p \in \mathcal{A}_p$. In the introduction we discussed that ρ being $\mathcal{A}_{p,q}$ implies that $(\rho(x))^p \in \mathcal{A}_p$ and $(\rho^*(x))^q \in \mathcal{A}_q$, both uniformly independent of the choice of unit vector x . Here we show this result by direct calculation, as in [8].

Lemma 1.5. *Let ρ be an $\mathcal{A}_{p,q}$ norm. Then $(\rho(x))^p \in \mathcal{A}_p$ and $(\rho^*(x))^q \in \mathcal{A}_q$ are both weights with the constant in the Muckenhoupt condition independent of the choice of unit vector x for all $x \neq 0$.*

Proof. Given an $\varepsilon > 0$ and nonzero vector x choose a unit vector $y(x, \varepsilon)$ such that

$$\langle \rho \rangle_{I,p} = \sup_y \frac{|(x, y)|}{\langle \rho \rangle_{I,p}^*(y)} \leq (1 + \varepsilon) \frac{|(x, y(x, \varepsilon))|}{\langle \rho \rangle_{I,p}^*(y(x, \varepsilon))}.$$

Thus

$$\begin{aligned} \left(\frac{1}{|I|} \int_I (\rho_t(x))^p \right)^{1/p} \left(\frac{1}{|I|} \int_I (\rho_t(x))^{-q} \right)^{1/q} &\leq \langle \rho \rangle_{I,p}(x) \langle \frac{\rho_t^*(y(x, \varepsilon))}{|(x, y(x, \varepsilon))|} \rangle_{I,q} \\ &\leq \frac{\langle \rho \rangle_{I,q}(x) \langle \rho^* \rangle_{I,q}(y(x, \varepsilon))}{|(x, y(x, \varepsilon))|} \leq C \frac{\langle \rho \rangle_{I,p}(x) \langle \rho \rangle_{I,q}^*(y(x, \varepsilon))}{|(x, y(x, \varepsilon))|} \leq (1 + \varepsilon) C \end{aligned}$$

In the first line we use (1.3) and in the last line we use Lemma 1.4. By letting ε decrease to 0 we see that the same constant from the $\mathcal{A}_{p,q}$ condition may be used in scalar \mathcal{A}_p condition. The proof that $(\rho^*(x))^q \in \mathcal{A}_q$ is similar. \square

1.4. Comparing matrix norms and arbitrary norms on a vector space. Every invertible positive matrix W yields a norm on a finite dimensional vector space via

$$\|x\|_W^2 = (Wx, x) = \|W^{1/2}x\|^2.$$

A useful result, John's theorem, discussed and used in [2], is that for an arbitrary norm ρ there is a matrix W such that

$$C_n^{-1}\rho(x) \leq \|W^{1/2}x\| \leq C_n\rho(x),$$

where C_n only depends on the dimension of the vector space.

Because of this inequality it is often possible to prove a theorem for matrix weights and use John's theorem to get an easy generalization to arbitrary norms.

2. SCALAR MUCKENHOUPPT CONDITION IMPLIES THE VECTOR ONE UNDER $\mathcal{A}_{0,0}$ CONDITION

Conceptually, the obstruction to $(\rho(x))^p \in A_p$ and $(\rho^*(x))^q \in A_q$ implying that $\rho \in \mathcal{A}_{p,q}$ comes from the fact that neither of these conditions can control how much the norms can be rotating as t varies.

It turns out that the $\mathcal{A}_{0,0}$ condition mentioned above as the weakest $\mathcal{A}_{p,q}$ condition controls this rotation.

Let us first analyze $\mathcal{A}_{0,0}$ condition

$$\langle \rho^* \rangle_{I,0} \leq C \langle \rho \rangle_{I,0}^*$$

First of all notice that the inverse inequality $\langle \rho \rangle_{I,0}^* \leq \langle \rho^* \rangle_{I,0}$ is trivial. Indeed, by the definition of dual norm

$$|(x, y)| \leq \rho_t(x) \rho_t^*(y) \quad \forall t.$$

Taking logarithms and averaging over I we get

$$\ln |(x, y)| \leq \int_I \ln(\rho_t(x)) dt + \int_I \ln(\rho_t^*(y)) dt$$

which implies

$$\frac{|(x, y)|}{\langle \rho(x) \rangle_{I,0}} \leq \langle \rho^*(y) \rangle_{I,0}.$$

Taking the supremum over all $x \neq 0$ we get the desired inequality.

While the average $\langle \rho^* \rangle_{I,0}$ is not generally a norm, only a positive 1-homogeneous function, the $\mathcal{A}_{0,0}$ condition implies that it is equivalent to the norm $\langle \rho \rangle_{I,0}^*$.

The next observation is that $\mathcal{A}_{0,0}$ condition is apparently non-symmetric: if $\rho \in \mathcal{A}_{0,0}$ then taking dual norms we get

$$\langle \rho \rangle_{I,0}^{**} \leq C \langle \rho^* \rangle_{I,0}^*,$$

which is weaker than $\rho^* \in \mathcal{A}_{0,0}$.

The theorem below shows that if the scalar weights $t \mapsto \rho_t(x)^p$, $t \mapsto \rho_t^*(x)$ are uniformly A_p and A_q weights respectively, and $\rho \in \mathcal{A}_{0,0}$, then ρ is an $\mathcal{A}_{p,q}$ weight. In fact, one can even replace A_p and A_q by the weaker A_∞ condition.

Theorem 2.1. *Let the weights $w_x, v_x, w_x(t) := (\rho_t(x))^p, v_x(t) := (\rho_t^*(x))^q$, $1/p + 1/q = 1$ be uniformly (in $x \in \mathbb{R}^n$) A_∞ , and let ρ be $\mathcal{A}_{0,0}$. Then the weight ρ satisfies $\mathcal{A}_{p,q}$ condition.*

Proof. Using the A_∞ condition for v_x (i.e. $\mathcal{A}_{q,0}$ condition for the norm-valued function $t \mapsto \rho_t^*(x)$ on a one-dimensional space) we get

$$\langle \rho^*(x) \rangle_{I,q} \leq C \langle \rho^*(x) \rangle_{I,0} \leq C_1 \langle \rho(x) \rangle_{I,0}^*$$

where the last inequality comes from the $\mathcal{A}_{0,0}$ condition. Using the A_∞ condition on w_x (but not using $\mathcal{A}_{0,0}$ condition for ρ^*) we may conclude that

$$\langle \rho(x) \rangle_{I,p} \leq C \langle \rho(x) \rangle_{I,0}$$

and taking the dual gives us

$$\langle \rho(x) \rangle_{I,0}^* \leq C \langle \rho(x) \rangle_{I,p}^*.$$

When this is combined with the first inequality in this proof we attain

$$\langle \rho^*(x) \rangle_{I,q} \leq C \langle \rho(x) \rangle_{I,p}^*$$

which is the $\mathcal{A}_{p,q}$ condition. \square

Note, that in the theorem we only need the condition $\rho \in \mathcal{A}_{0,0}$ and we did not need the ‘‘symmetric’’ condition $\rho^* \in \mathcal{A}_{0,0}$. Of course, the same theorem would be true under the assumption that only ρ^* is an $\mathcal{A}_{0,0}$ weight.

3. SCALAR MUCKENHOUPT CONDITION IMPLIES THE VECTOR ONE IN DIMENSION 2

While the statement ‘‘Uniformly A_2 implies $\mathcal{A}_{2,2}$.’’ is generally false (as we will demonstrate later in the paper), if we restrict our attention to weights on \mathbb{R}^2 we can show that this statement is true. The reason that this is true is that the exponent in the A_2 condition interacts in a nice way with the exponents used to compute an inverse volume in dimension 2.

To simplify calculations we will perform all calculations with matrix-valued weights, and later approximate arbitrary weights with matrix-valued weights.

Theorem 3.1. *Let W be a 2×2 positive matrix valued function such that the scalar weights $(W(\cdot)x, x)$ and $(W^{-1}(\cdot)x, x)$ are uniformly A_2 over all unit vectors $x \in \mathbb{R}^2$. Then W satisfies the matrix \mathbf{A}_2 condition.*

3.1. Proof of Theorem 3.1.

3.1.1. *Volumes, determinants, and norms.* We will now discuss volumes of unit balls for various norms, since determinants will be key to our understanding of the connection between $\mathcal{A}_{2,2}$ and A_2 conditions when the vector space being considered has real dimension 2.

For a norm $\boldsymbol{\rho}$ on \mathbb{R}^n the volume $v(\boldsymbol{\rho})$ of its unit ball $\{x \in \mathbb{R}^n : \boldsymbol{\rho}(x) \leq 1\}$ can be computed by

$$v(\boldsymbol{\rho}) = \frac{1}{n} \int_{S_{n-1}} \boldsymbol{\rho}(x) d\sigma_{n-1}(x),$$

where $S_{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is the unit sphere in \mathbb{R}^n (with respect to the standard norm), σ_{n-1} is the surface measure on S_{n-1} and C_n is the constant depending on the dimension.

For a norm-valued weight ρ let us compute the volume (of the unit ball) of the average norm $\langle \rho \rangle_{I,2}$:

$$(3.1) \quad v(\langle \rho \rangle_{I,2}) = \frac{1}{n} \int_{S_{n-1}} \left(\int_I \rho_t(x)^2 dt \right)^{-n/2} d\sigma_{n-1}(x).$$

On the other hand, computing the average of the volumes $v(\rho_t)$ yields

$$\langle v(\rho) \rangle_I = \frac{1}{n} \int_I \int_{S_{n-1}} (\rho_t(x))^{-n} d\sigma_{n-1}(x) dt.$$

By changing the order of integration we attain

$$(3.2) \quad \langle v(\rho) \rangle_I = \frac{1}{n} \int_{S_{n-1}} \int_I (\rho_t(x))^{-n} dt d\sigma_{n-1}(x).$$

We note that if $n = 2$ then (3.1) and (3.2) become

$$(3.3) \quad v(\langle \rho \rangle_{I,2}) = \frac{1}{2} \int_{S_1} \left(\int_I \rho_t(x)^2 dt \right)^{-1} d\sigma_1(x)$$

$$(3.4) \quad \langle v(\rho) \rangle_I = \frac{1}{2} \int_{S_1} \int_I (\rho_t(x))^{-2} dt d\sigma_1(x)$$

If we assume that the scalar weights $t \mapsto \rho_t(x)^2$ have the A_2 norm bounded by C uniformly in nonzero x , we have:

$$\int_I \rho_t(x)^{-2} dt \leq C \left(\int_I \rho_t(x)^2 dt \right)^{-1}$$

which together with (3.3), (3.4) implies

$$(3.5) \quad \langle v(\rho) \rangle_I \leq C v(\langle \rho \rangle_{I,2}).$$

The following well-known lemma reminds the reader how volumes and determinants are related.

Lemma 3.2. *Let $\boldsymbol{\rho}$ be a norm in \mathbb{R}^n given by $\boldsymbol{\rho}(x) = \|A^{1/2}x\|$, where A be a positive definite matrix. Then*

$$v(\boldsymbol{\rho}) = C_n \det A^{-1/2},$$

where C_n is a constant depending on the dimension n (volume of the unit ball in \mathbb{R}^n).

Let W be the weight from Theorem 3.1. For the norm-valued weight ρ given by $\rho_t(x) = \|W^{1/2}(t)x\|$ (so $\langle \rho \rangle_{I,2}(x) = \|\langle W \rangle_I^{1/2}x\|$) equation (3.5) becomes

$$\langle \det W^{-1/2} \rangle_I \leq C \det \langle W \rangle_I^{-1/2}.$$

Similarly, (3.5) applied to the weight ρ , $\rho_t(x) = \|W^{-1/2}(t)x\|$ gives us

$$\langle \det W^{1/2} \rangle_I \leq C \det \langle W^{-1} \rangle_I^{-1/2}$$

Multiplying two above inequalities and using the fact that $1 \leq f_I \varphi f_I \varphi^{-1}$ we get

$$1 \leq \langle \det W^{-1/2} \rangle_I \langle \det W^{1/2} \rangle_I \leq C \det \langle W \rangle_I^{-1/2} \det \langle W^{-1} \rangle_I^{-1/2},$$

which implies

$$(3.6) \quad \det \langle W \rangle_I \det \langle W^{-1} \rangle_I \leq C.$$

But this inequality is an equivalent version of the matrix \mathbf{A}_2 condition, as the reasoning below asserts. \square

3.1.2. Determinant version of the matrix \mathbf{A}_2 condition. The fact that condition (3.6) is equivalent to the matrix Muckenhoupt \mathbf{A}_2 condition is well-known. It is implicitly contained in [7] and [5], and it was first explicitly stated and proved in [1]. We present the reasoning below just for the sake of completeness.

Recall that the matrix \mathbf{A}_2 condition

$$\|\langle W^{-1} \rangle_I^{1/2} \langle W \rangle_I^{1/2}\| \leq C,$$

can be rewritten as

$$\|\langle W \rangle_I^{1/2} \langle W^{-1} \rangle_I \langle W \rangle_I^{1/2}\| \leq C^2$$

Note that

$$\det(\langle W \rangle_I^{1/2} \langle W^{-1} \rangle_I \langle W \rangle_I^{1/2}) = \det \langle W \rangle_I \det \langle W^{-1} \rangle_I.$$

Then the equivalence of (3.6) and the matrix \mathbf{A}_2 condition follows from the two simple lemmas below (in both lemmas the bold symbol \mathbf{I} is used for the identity matrix, not to be confused with the interval I).

The first lemma is a standard and well-known fact.

Lemma 3.3. *Let $A = A^*$ be an $n \times n$ matrix satisfying $A \geq \mathbf{I}$. Then $\det A \leq \|A\|^n$.*

Proof. Trivial. \square

The next lemma is also known, see for example Corollary 3.3 in [7].

Lemma 3.4. *Let W be a matrix-valued weight and let W^{-1} be integrable on an interval I . Then*

$$\langle W \rangle_I^{1/2} \langle W^{-1} \rangle_I \langle W \rangle_I^{1/2} \geq \mathbf{I}.$$

Proof. This lemma is simply a restatement of Lemma 1.4. Indeed, Lemma 1.4 applied to the weight ρ given by $\rho_t(x) = \|W^{1/2}(t)x\|$ gives us

$$\langle W^{-1} \rangle_I \geq \langle W \rangle_I^{-1}$$

Left and right multiplying both sides of the inequality by $\langle W \rangle_I^{1/2}$ we get the conclusion of the lemma. \square

3.2. Theorem for norm-valued weights. Using John’s Theorem one can easily extend Theorem 3.1 to the case of norm-valued weights:

Theorem 3.5. *Let ρ be a weight on \mathbb{R}^2 so that the scalar weights $t \mapsto \rho_t(x)^2$ and $t \mapsto \rho_t^*(x)^2$ are uniformly A_2 for all vectors $x \neq 0$. Then ρ satisfies $\mathcal{A}_{2,2}$ condition.*

Proof. The main idea is to use John’s theorem to find a matrix-valued weight $W^{1/2}$ so that

$$\|W(t)^{1/2}x\| \leq \rho_t(x) \leq C\|W(t)^{1/2}x\|.$$

Then everything would follow immediately from Theorem 3.1. The only technical difficulty here is to guarantee that the function W is measurable.

There are several possible approaches to overcome this difficulty. One is to use theory of vector integration. Let us explain it in more details.

A norm ρ on \mathbb{R}^n is completely defined by its values on the standard unit sphere $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$ in \mathbb{R}^n , so one can interpret a norm-valued function ρ as a function with values in $C(S)$.

We assumed that the scalar functions $t \mapsto \rho_t(x)$ are measurable for all x . Let us show that the function $\rho : \mathbb{R} \rightarrow C(S)$ is strongly (Bochner) measurable, i.e. that it can be represented as a uniform limit of measurable functions with countably many values. Since the space $C(S)$ is separable, by the Gelfand–Pettis theorem (see [4], Theorem 3.2.2) it is sufficient to check that the function ρ is weakly measurable, i.e. that the scalar-valued functions $t \mapsto \langle \rho_t, \mu \rangle$ are measurable for all $\mu \in C(S)^*$.

Given a finite Borel measure μ and a continuous function φ on S , and a partition $\mathcal{P} = \{E_k \ni x_k, k = 1, 2, \dots, m\}$ of S into finitely many Borel sets E_k one can define a “Riemann sum”

$$\Sigma_{\mathcal{P}}(\varphi, \mu) := \sum_k \mu(E_k) \varphi(x_k)$$

It is an easy exercise to show that for any sequence of partitions \mathcal{P}_k such that the maximal diameter of the elements of \mathcal{P}_k tends to 0 as $k \rightarrow \infty$

$$\Sigma_{\mathcal{P}_k}(\varphi, \mu) \rightarrow \int_S \varphi d\mu \quad \text{as } k \rightarrow \infty.$$

Let us pick such a sequence of partitions \mathcal{P}_k . The measurability of $t \mapsto \rho_t(x)$ implies that the functions $t \mapsto \sum_{\mathcal{P}_k}(\rho_t, \mu)$ are measurable, and so is the limit function $t \mapsto \int_S \rho_t(x) d\mu(x)$.

So we have proved that the function $\rho : \mathbb{R} \rightarrow C(S)$ is strongly measurable, i.e. that ρ is a uniform limit of a sequence of functions $\varphi_k(t) = \sum_j a_j^k \chi_{A_j^k}(t)$, where $a_j^k \in C(S)$ and A_j^k are Borel subsets of \mathbb{R} .

A priori the functions a_j^k do not need to be norms, but without loss of generality one can always replace a_j^k by $\rho_{t_j^k}$, where t_j^k is an arbitrary point in A_j^k . Therefore, ρ is a uniform limit of measurable norm-valued functions ρ^k taking countably many values.

Applying John's Theorem to ρ^k one gets the matrix-valued functions W_k such that

$$\frac{1}{C} \|W_k^{1/2}(t)x\| \leq \rho_t^k(x) \leq C \|W_k^{1/2}(t)x\|$$

If

$$\frac{1}{K} \|x\| \leq \rho_t(x) \leq K \|x\|, \quad \forall t \quad \forall x \in \mathbb{R}^n,$$

then the weight W_k with sufficiently large k approximates ρ with slightly worse constant C .

For the general case a slightly more elaborate reasoning is needed. Namely, consider the functions

$$M(t) = \sup\{\rho_t(x) : \|x\| = 1\}, \quad m(t) = \inf\{\rho_t(x) : \|x\| = 1\},$$

and define the sets

$$A_k := \{t \in \mathbb{R} : m(t) > 2^{-k}, M(t) < 2^k\}.$$

Clearly $A_k \subset A_{k+1}$ and the set $\mathbb{R} \setminus \cup_k A_k$ has zero Lebesgue measure.

On each set $A_{k+1} \setminus A_k$ the norm-valued function ρ (restricted to the unit sphere) is bounded away from 0 and ∞ , so we can apply the above reasoning to approximate ρ by measurable weights on each of the sets $A_{k+1} \setminus A_k$ separately. \square

4. A PRELIMINARY COUNTEREXAMPLE FOR 6×6 MATRICES ON A CUBE IN \mathbb{R}^{15}

In this section we will construct a preliminary counterexample to the statement "The scalar weights (Wx, x) , $(W^{-1}x, x)$ being uniformly A_2 implies that W satisfies the matrix \mathbf{A}_2 condition." To do this we will have to consider higher dimensional domains. We will extend the definitions of various Muckenhoupt conditions by taking supremums over cubes or balls instead of intervals.²

Namely, we will construct the family W_ε , $\varepsilon > 0$ of 6×6 continuous weights on a cube $Q = [0, 1]^{15} \subset \mathbb{R}^{15}$ such that

²There are interesting issues that arise when sets with varying geometry, such as arbitrarily "skinny" rectangles are considered. We do not make use of these here.

- (1) The weights W_ε are not uniformly matrix \mathbf{A}_2 , namely.

$$\left\| \left(\int_Q W_\varepsilon \right)^{1/2} \left(\int_Q W_\varepsilon^{-1} \right)^{1/2} \right\| \geq C/\sqrt{\varepsilon}$$

- (2) The scalar weights $w_{\varepsilon,x} := (W_\varepsilon x, x)$ and $v_{\varepsilon,x} := (W_\varepsilon^{-1} x, x)$ nevertheless satisfy the Muckenhoupt condition A_2 and their Muckenhoupt norms are uniformly bounded in ε , i.e. for any 15-dimensional cube $R \subset Q$

$$\left(\int_R w_{\varepsilon,x} \right)^{1/2} \left(\int_R w_{\varepsilon,x}^{-1} \right) \leq C < \infty,$$

uniformly in $0 < \varepsilon < 2^{-10}$ (or any small number) and the same for $v_{\varepsilon,x}$.

This family of weights will later be (Section 4.2) used to construct a real counterexample: we will use a space filling curve to transfer the weights to an interval, and then use these transferred weights to construct one weight on the whole real line

The main idea of the counterexample is to take for the weight rotations of an orthogonal projection. If there are enough “degrees of freedom”, it will easy to make the scalar weights $w_x = (Wx, x)$ to be Muckenhoupt. However, the “rotating projection” counterexample does not work, because the inverse weight W^{-1} is not defined in this case. Therefore, the statement above, and the construction below are a bit more complicated.

Before proceeding further we present an easy lemma that will help us estimate the norms of the product of positive matrices.

Lemma 4.1. *Let $A > C_1 \mathbf{I}$ and $B > C_2 \mathbf{I}$ be positive definite matrices with $C_1, C_2 > 0$ being constants. Then $\|A^{1/2} B^{1/2}\| \geq \sqrt{C_1 C_2}$.*

Proof. The assumption $A > C_1 \mathbf{I}$ implies

$$\|A^{1/2} x\|^2 = (Ax, x) \geq C_1 \|x\|^2 \quad \forall x,$$

therefore $\|A^{1/2} x\| \geq \sqrt{C_1} \|x\|$ for all x . Similarly, $\|B^{1/2} x\| \geq \sqrt{C_2} \|x\|$ for all x , so

$$\|A^{1/2} B^{1/2} x\| \geq \sqrt{C_1} \|B^{1/2} x\| \geq \sqrt{C_1} \sqrt{C_2} \|x\|, \quad \forall x,$$

which immediately implies the conclusion of the lemma. \square

4.1. Matrix weights on SO_6 . Temporarily let the domain be SO_6 , the Lie group of all rotations in \mathbb{R}^6 , which has real dimension 15. Instead of computing the Muckenhoupt conditions by averaging over intervals, we will compute them by averaging over balls in the natural metric on SO_6 . One can simply think, for example that the measure on SO_6 as being 15 dimensional Hausdorff measure on a subset of \mathbb{R}^{36} .

Let P and Q be complimentary orthogonal projection onto first three and last three coordinates in \mathbb{R}^6

Define the weight W_ε on SO_6 for use for our counterexample by

$$(4.1) \quad W_\varepsilon(U) = U^{-1}(P + \varepsilon Q)U, \quad U \in SO_6.$$

We will show that for any $\varepsilon > 0$ and any $x \in \mathbb{R}^6$ the weights $w_{\varepsilon,x} = (W_\varepsilon x, x)$ $v_{\varepsilon,x} = (W_\varepsilon^{-1}x, x)$ are uniformly (in ε and x) A_2 weights. Since $\varepsilon I \leq W_\varepsilon \leq I$, one can conclude that each weight W_ε is an A_2 weight, but we will show that the constants blow up when $\varepsilon \rightarrow 0$.

4.1.1. *Why the weights W_ε are not uniformly A_2 .* To show that the constants in A_2 conditions blow up, consider a measurable $\mathcal{U} \subset SO_6$ of positive measure. We want to estimate the average

$$\int_{\mathcal{U}} W_\varepsilon(U) dH_{15}(U).$$

where H_{15} is the 15-dimensional Hausdorff measure on SO_6 .

We first estimate the average on the vector \mathbf{e}_1 —the first vector in the standard basis in \mathbb{R}^6 . If we get the estimate that depends only on the measure $|\mathcal{U}|$ (we use symbol $|\mathcal{U}|$ for $H_{15}(\mathcal{U})$), the invariance with respect to the group action gives the estimate for all unit vectors.

Note that

$$(W_\varepsilon(U)\mathbf{e}_1, \mathbf{e}_1) \geq (U^{-1}PU\mathbf{e}_1, \mathbf{e}_1) = \|PU\mathbf{e}_1\|^2.$$

For $0 < \alpha \leq 1$ let $\mathcal{U}_\alpha := \{U \in SO_6 : \|PU\mathbf{e}_1\| < \alpha\}$. Since $|\mathcal{U}_\alpha| \rightarrow 0$ as $\alpha \rightarrow 0$, we can pick $\alpha = \alpha(|\mathcal{U}|)$ such that $|\mathcal{U}_\alpha| < |\mathcal{U}|/2$. Note that for $U \in \mathcal{U} \setminus \mathcal{U}_\alpha$ we have $\|PU\mathbf{e}_1\| \geq \alpha$. Taking into account that $|\mathcal{U} \setminus \mathcal{U}_\alpha| \geq |\mathcal{U}|/2$ we get

$$\int_{\mathcal{U}} (W_\varepsilon(U)\mathbf{e}_1, \mathbf{e}_1) dH_{15}(U) \geq \alpha^2/2.$$

The invariance with respect to the group action implies

$$\int_{\mathcal{U}} W_\varepsilon(U) dH_{15}(U) \geq c(|\mathcal{U}|)\mathbf{I}$$

where $c(|\mathcal{U}|) = \alpha(|\mathcal{U}|)^2/2$.

To estimate the average of W_ε^{-1} let us notice that

$$(4.2) \quad W_\varepsilon^{-1}(U) = U^{-1}(P + \varepsilon^{-1}Q)U = \varepsilon^{-1}U^{-1}(Q + \varepsilon P)U = \frac{1}{\varepsilon}W_\varepsilon(U_0U)$$

where U_0 is any transformation in SO_6 interchanging the ranges of P and Q . Therefore, the computation of the average of W_ε^{-1} can be reduced to the average of W_ε :

$$\int_{\mathcal{U}} W_\varepsilon^{-1} dH_{15} = \frac{1}{\varepsilon} \int_{U_0^{-1}\mathcal{U}} W_\varepsilon(U_0U) dH_{15}(U) \geq \frac{c(|U_0^{-1}\mathcal{U}|)}{\varepsilon} \mathbf{I} = \frac{c(|\mathcal{U}|)}{\varepsilon} \mathbf{I}.$$

Therefore by Lemma 4.1

$$(4.3) \quad \left\| \left(\int_{\mathcal{U}} W_\varepsilon \right)^{1/2} \left(\int_{\mathcal{U}} W_\varepsilon^{-1} \right)^{1/2} \right\| \geq c(|\mathcal{U}|)/\sqrt{\varepsilon}.$$

So, the weights W_ε are not uniformly A_2 .

4.1.2. *Why the scalar weights are uniformly A_2 .* Now we must show that the weights $w_{\varepsilon,x}$, $w_{\varepsilon,x}(U) := (W_\varepsilon(U)x, x)$ are uniformly (in $\varepsilon > 0$ and $x \in \mathbb{R}^6 \setminus \{0\}$) A_2 weights. Note, that because of translation invariance, it is again sufficient to consider only the case $x = \mathbf{e}_1$, \mathbf{e}_1 is the first vector in the standard basis in \mathbb{R}^6 .

First of all, consider the case $\varepsilon = 0$. For the weight $w := w_{0,\mathbf{e}_1}$ we have $w(U) = \|PU\mathbf{e}_1\|^2$. It is easy to see that the set $M := \{U \in SO_6 : PU\mathbf{e}_1 = 0\}$ is a smooth submanifold of SO_6 of dimension 12.

Indeed, $PU\mathbf{e}_1 = 0$ means that the first column of U belongs to the 3-dimensional space $\text{Ker } P$. But the first column must be normalized, so all possible first columns form a manifold of dimension 2. And it is an easy exercise (left to the reader) to show that the set of all matrices U with a given first column can be parametrized by the group SO_5 , which has dimension 10. Namely, for a matrix $U_0 \in SO_6$ the set of all matrices $U \in SO_6$ with the same first column can be parametrized as

$$U = U_0 \begin{pmatrix} 1 & 0 \\ 0 & \tilde{U} \end{pmatrix}, \quad \tilde{U} \in SO_5.$$

Combining all together we get $\dim M = 10 + 2 = 12$.

Lemma 4.2. *For $U \in SO_6$ we have $\text{dist}(U, M) \asymp \|PU\mathbf{e}_1\|$, where \asymp means equivalence in the sense of two sided estimate.*

Proof. The inequality $\|U\mathbf{e}_1\| \leq \text{dist}(U, M)$ is trivial.

To prove the opposite inequality consider consider $x = U\mathbf{e}_1$. One can easily find $V \in SO_6$ such that $Vx \in \text{Ker } P = \text{Ran } Q$ and $\|I - V\| \leq C\|Px\|$: one can pick for such V a matrix which is the appropriate rotation in the plane $\text{span}(x, Qx)$ and is identity in the the $\text{span}(x, Qx)^\perp$.

Then clearly $VU \in M$ and $\|U - VU\| \leq C\|Px\| = C\|PU\mathbf{e}_1\|$.

From this fact, recalling the definition of w we get that

$$w(U) \asymp [\text{dist}(U, M)]^2.$$

The submanifold M of SO_6 has codimension 3, so it is easy to see that the weight w satisfies the A_2 condition.

Indeed, it is well known and is easy to check directly that the weight on \mathbb{R}^n that behaves as $|x|^p$, $|p| < n$ around 0 and regular (bounded away from zero and infinity) outside a neighborhood of 0 satisfies A_2 condition.

The same computations show that the weight behaving like $[\text{dist}(x, M)]^p$ (where M is a linear subspace of *codimension* n and $|p| < n$) near M and regular everywhere else satisfies the Muckenhoupt condition A_2 . The standard argument shows that the conclusion remains true if one takes for M a smooth compact manifold of codimension n , which is exactly our case.

Of course, one can object that our M is a submanifold of SO_6 , not of \mathbb{R}^d . However, because of the compactness of SO_6 one only needs to check

the Muckenhoupt condition on small balls, and locally there is no difference between SO_6 and \mathbb{R}^{15} , so everything works in our case. \square

We now need to show that the weights $w_\varepsilon := w_{\varepsilon, e_1}$ are uniformly A_2 weights. The condition (4.1) and the definition of w_ε imply that

$$w_\varepsilon = \varepsilon + (1 - \varepsilon)w.$$

And from here one can easily conclude that all the weights w_ε satisfy the Muckenhoupt condition A_2 and their Muckenhoupt norms are not worse than $[w]_{A_2}$. This can be proved by a simple direct computation, but we present another, more “high brow” explanation. Namely, let us consider the 2×2 matrix weight

$$V = \begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix}$$

which clearly satisfies the matrix \mathbf{A}_2 condition. Then $w_\varepsilon = (W a_\varepsilon, a_\varepsilon)$, where $a_\varepsilon = (\sqrt{1 - \varepsilon}, \sqrt{\varepsilon})^T$, so (see the discussion in the end of Section 0 or Lemma 1.5) the weights w_ε are uniformly A_2

And as we already discussed above, the translation invariance (with respect to group action) gives us the same conclusion for the weights $w_{\varepsilon, x}$, $x \in \mathbb{R}^6$.

Finally, we need to show that the weight $v_{\varepsilon, x} = (W_\varepsilon^{-1}x, x)$ are also uniformly A_2 weights. But it follows from (4.2) that

$$v_{\varepsilon, x}(U) = \varepsilon^{-1}w_{\varepsilon, x}(U_0U), \quad U \in SO_6,$$

and since the translation and multiplication by a constant do not change the Muckenhoupt norm of a weight, the statement about $v_{\varepsilon, x}$ follows from the one about $w_{\varepsilon, x}$.

4.2. Making the domain be a cube in \mathbb{R}^{15} . Consider some chart, i.e. a smooth injective mapping $\varphi : Q \rightarrow SO_6$ (where Q is a cube in \mathbb{R}^{15}) such, that the derivative φ' and its inverse $[\varphi']^{-1}$ are uniformly bounded on Q . We assume that the cube Q is a closed cube, and φ is actually defined in some neighborhood of Q .

Using translations and dilations we can assume without loss of generality that $Q = [0, 1]^{15}$. Define the 6×6 matrix weights $\widetilde{W}_\varepsilon$ on Q by $\widetilde{W}_\varepsilon := W_\varepsilon \circ \varphi$.

The set $\mathcal{U} := \varphi(Q)$ has non-empty interior, so (4.3) implies

$$\left\| \left(\int_Q \widetilde{W}_\varepsilon \right)^{1/2} \left(\int_Q \widetilde{W}_\varepsilon^{-1} \right)^{1/2} \right\| \geq c(|\mathcal{U}|)/\sqrt{\varepsilon}$$

(of course here one needs the fact that $[\varphi']^{-1}$ is uniformly bounded on Q).

On the other hand, it is easy to see that the change of variables φ with uniformly bounded φ' and $[\varphi']^{-1}$ preserves the Muckenhoupt A_2 condition with the control on the Muckenhoupt norm, so the weights $w_{\varepsilon, x}(\cdot) := (W_\varepsilon(\cdot)x, x)$, $v_{\varepsilon, x}(\cdot) := (W_\varepsilon^{-1}(\cdot)x, x)$ satisfy condition 2 from the beginning of Section 4

So renaming the weights $\widetilde{W}_\varepsilon$ to W_ε we get the desired family of weights.

5. TRANSFERRING COUNTEREXAMPLE TO THE REAL LINE.

In this section we first, using a space filling curve, transfer the constructed above family of weights W_ε on a 15-dimensional cube to an interval. Then we will use this family to construct the real counterexample.

The main idea of this section is that given a carefully constructed space-filling curve $\gamma : [0, 1] \mapsto [0, 1]^{15}$ and a measurable function $g : [0, 1]^{15} \mapsto \mathcal{E}$ where \mathcal{E} is some vector space, the function $f(\cdot) := g(\gamma(\cdot))$ will have averaging properties on intervals similar to those of g on 15-cubes. We will use this idea to define $\widetilde{W}_\varepsilon(\cdot) = W_\varepsilon(\gamma(\cdot))$. The W_ε are the matrix-valued functions defined on $[0, 1]^{15}$ which demonstrated that a uniform \mathcal{A}_2 condition on W and W^{-1} would not allow us to control the \mathcal{A}_2 norm of W . Thus $\widetilde{W}_\varepsilon$ would provide an example of matrix-valued functions with the same properties except the domain would be $[0, 1]$.

5.1. Defining a space-filling curve. We begin with an example of a curve that maps an interval onto a square, and then work up to an example of a curve that maps an interval to a 15-cube. A more complete discussion of space-filling curves can be found in Sagan [6].

We recall the standard construction of the square-filling Peano curve. Define $\gamma_1 : [0, 1] \mapsto [0, 1] \times [0, 1] = [0, 1]^2$ by $\gamma(t) = (t, t)$. Then form the piecewise affine function γ_2 by dividing the square into nine subsquares and mapping a ninth of the unit interval to each square, as it is shown in the picture below. Note that both γ_1 and γ_2 have the same starting and end points.

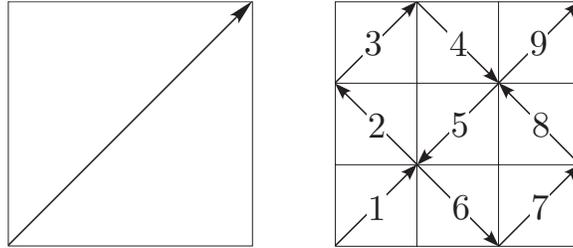


FIGURE 1. Functions γ_1 (left) and γ_2 (right). Numbers indicate which subinterval of I corresponds to which affine part of γ_2

To get the function γ_3 we replace each affine piece of γ_2 by a piecewise affine function, using the same pattern we used to get from γ_1 to γ_2 . This is probably clearer pictorially, see Fig. 2 below.

Finally we define $\gamma(t) = \lim_{n \rightarrow \infty} \gamma_n(t)$. The map $\gamma : [0, 1] \mapsto [0, 1]^2$ has the following properties:

- γ is onto and continuous
- For any 9-adic interval $I = [m3^{-2k}, (m + 1)3^{-2k}]$, its image $\gamma(I)$ is a square with area 3^{-2k} .

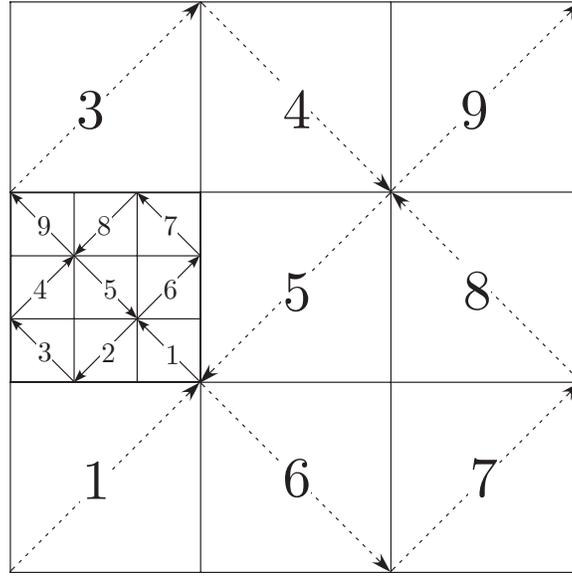


FIGURE 2. How to obtain γ_3 from γ_2 : the second affine segment of γ_2 is replaced by piecewise affine function. Of course, all other large segments should be replaced in the same fashion by piecewise affine functions

- Any two consecutive 9-adic intervals of the same size are mapped to adjacent squares.

Let us generalize this construction to higher dimensions. Consider first the case of dimension 3. We start by defining the first function γ_1 by $\gamma_1(t) := (t, t, t)$. Then we need to split the cube into 27 triadic cubes and to construct a continuous piecewise affine function γ_2 , which will serve as a pattern for replacing affine pieces by a finer piecewise affine function in the construction. We will need to construct a function γ_2 such that

- γ_2 is a piecewise affine continuous function; its domain is divided into 27 equal intervals, and the function is affine on each such an interval;
- $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$, i.e. γ_1 and γ_2 have the same starting point and the same end point;
- the image of each affine piece of γ_2 is a diagonal of a triadic cube;
- moreover, γ_2 passes (but only once) through each of the 27 triadic cubes.

There are many ways to construct the curve, and we present what seems to us to be the simplest one, utilizing two-dimensional construction discussed above. Namely, we split the cube into 3 “layers”, $0 \leq x_3 \leq 1/3$, $1/3 \leq x_3 \leq 2/3$, $2/3 \leq x_3 \leq 1$, and γ_2 will go first through the first layer, then through the second, and finally through the last.

Let us consider the first layer: the first 2 coordinates of γ_2 are given exactly by the two-dimensional version of the function γ_2 constructed above. As for the last coordinate x_3 , on each of the small intervals it is either affinely increasing from 0 to $1/3$ or decreasing from $1/3$ to 0. On the Fig. 3 below solid arrows correspond to the segments where x_3 increases, and the dashed ones to the segments where x_3 decreases. Note, that we started at the “bottom” of the first “layer” and ended up on its top, in the opposite corner.

On the second “layer” we work our way back to the original corner, and on the third “layer” we essentially repeat the first one. We believe that the construction of γ_2 should be quite evident from Fig. 3 below.

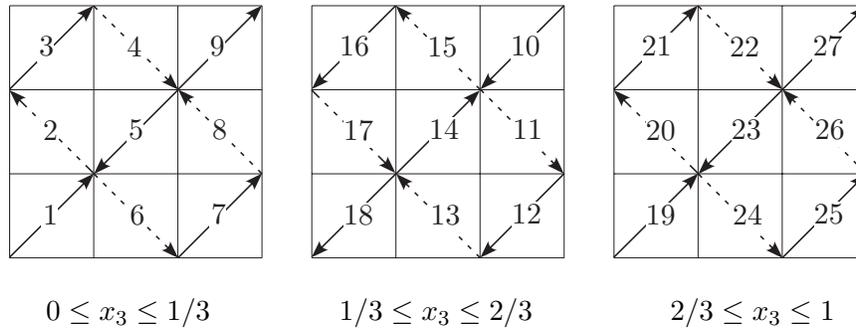


FIGURE 3. Function γ_2 in dimension 3. Solid arrows are going “up” (the last coordinate x_3 increases), and the dashed ones are going “down”.

Again, by iterating the process of replacing a diagonal segment with a small copy of γ_2 we obtain a sequence $\{\gamma_n\}$ and define the cube filling function $\gamma := \lim_{n \rightarrow \infty} \gamma_n$.

Now it is easy to generalize the construction to arbitrary dimension d . Again, we start with the function γ_1 , $\gamma_1(t) = (t, t, \dots, t)$. To construct the function γ_2 (and thus the replacement pattern) we split the cube into 3 layers, using the last coordinate x_d . On each layer the first $d - 1$ coordinates of γ can be obtained from the function γ_2 in the previous dimension, and on each segment the last coordinate either increases or decreases affinely. Note, that since each layer has an odd number of triadic cubes, we always start from the bottom of a layer and end up at its top, so we can join pieces for each layer into a continuous function.

Again, replacing inductively each affine segment by a piecewise affine function we get a sequence γ_n and the d -cube filling curve $\gamma = \lim_{n \rightarrow \infty} \gamma_n$.

We need the function in the dimension 15. Note, that it has the following properties: $\gamma : [0, 1] \mapsto [0, 1]^{15}$ such that

- $\gamma : [0, 1] \mapsto [0, 1]^{15}$ is continuous and onto;

- for any 3^{15} -adic interval $I = [m3^{-15k}, (m+1)3^{-15k}]$ (which has length 3^{-15k}) its image $\gamma(I)$ is a 15-cube with volume 3^{-15k} .
- Any two consecutive 3^{15} -adic intervals of the same size are mapped to adjacent 3-adic cubes.

Approximating integral by the Riemann sums we get the following

Lemma 5.1. *Let f and g be continuous functions on $[0, 1]$ and $[0, 1]^{15}$ respectively. Let us further suppose that $f = g \circ \gamma$, where $\gamma : [0, 1] \mapsto [0, 1]^{15}$ is as above. Then*

$$\int_{\gamma(I)} g(s) ds = \int_I f(t) dt$$

for all intervals I .

Proof. The proof is trivial for 3^{15} -adic intervals, one need simply consider Riemann sums corresponding to partitions of I into smaller 3^{15} -adic intervals. The case of arbitrary interval I can be treated by approximation of I by a union of 3^{15} -adic intervals. \square

5.2. Preliminary counterexample on an interval. Let $W_\varepsilon : [0, 1]^{15} \mapsto M_{6 \times 6}$ be the family of matrix weights defined in subsection 4.2. Thus W_ε are positive matrix-valued functions such that W_ε has an \mathbf{A}_2 norm which blows up as $\varepsilon \searrow 0$, but the weights $w_{x,\varepsilon}(\cdot) := (W_\varepsilon(\cdot)x, x)$, $v_{x,\varepsilon}(\cdot) := (W_\varepsilon^{-1}(\cdot)x, x)$, are uniformly (in x and ε) scalar A_2 weights. Define $\widetilde{W}_\varepsilon : [0, 1] \mapsto M_{6 \times 6}$ by

$$(5.1) \quad \widetilde{W}_\varepsilon(s) = W_\varepsilon(\gamma(s)).$$

Considering averages over $[0, 1]$ and using Lemma 5.1 it is easy to see that the weights $\widetilde{W}_\varepsilon$ are not uniformly \mathbf{A}_2 weights.

We need to show that the scalar weights $\widetilde{w}_{x,\varepsilon}(\cdot) := (\widetilde{W}_\varepsilon(\cdot)x, x)$, $\widetilde{v}_{x,\varepsilon}(\cdot) := (\widetilde{W}_\varepsilon^{-1}(\cdot)x, x)$ are uniformly A_2 weights. Lemma 5.1 implies that the A_2 conditions holds uniformly on 3^{15} -adic intervals. To extend this to arbitrary intervals we use the following lemma:

Lemma 5.2. *Let g be an A_2 weight, $g : [0, 1]^{15} \mapsto [0, \infty)$. Let $f = g \circ \gamma : [0, 1] \mapsto [0, \infty)$. Then f is also an A_2 weight with*

$$[f]_{A_2} \leq (2^{15}3^{15})^2 [g]_{A_2}.$$

Proof. Given an interval $I \subset [0, 1]$, we have $3^{-15(m+1)} < |I| \leq 3^{-15m}$ for some integer m . Then I is in the union of (at most) two adjacent 3^{15} -adic intervals of length 3^{-15m} , so $\gamma(I)$ is in the union of at most 2 triadic cubes with side 3^{-m} . Then $\gamma(I) \subset Q$, where Q is a 15-cube, not necessarily triadic, with side $2 \cdot 3^{-m}$.

Then we use A_2 condition for the cube Q to get the estimate on I :

$$\begin{aligned} \left| \frac{1}{|I|^2} \int_I f(s) ds \int_I \frac{1}{f(s)} ds \right| &\leq \left| \frac{1}{|I|^2} \int_{\gamma(I)} g(t) dt \int_{\gamma(I)} \frac{1}{g(t)} dt \right| \leq \\ &\left| \frac{|Q|^2}{|I|^2 |Q|^2} \int_Q g(t) dt \int_Q \frac{1}{g(t)} dt \right| \leq \\ &\left(\frac{2^{15} 3^{-15(m)}}{3^{-15(m+1)}} \right)^2 [g]_{A_2} \leq (2^{15} 3^{15})^2 [g]_{A_2} \end{aligned}$$

□

5.3. Extending counterexample to \mathbb{R} . Now we are going to present the final counterexample, namely *one* weight W which fails to be an \mathbf{A}_2 weight, but such, that the scalar weights $(W(\cdot)x, x)$ and $(W^{-1}(\cdot)x, x)$ are uniformly A_2 weights.

Set $\varepsilon_k := 2^{-10} 2^{-k}$ and define $\widetilde{W}_k := \widetilde{W}_{\varepsilon_k}$. Let us now *symmetrize* the weights \widetilde{W}_k by defining the weights W_k on $[0, 1]$

$$(5.2) \quad W_k(t) = \begin{cases} \widetilde{W}_k(2t) & t < \frac{1}{2} \\ \widetilde{W}_k(2t-1) & t \geq \frac{1}{2}. \end{cases}$$

Since the initial weights \widetilde{W}_k were continuous on $[0, 1]$, the periodic extension of each W_k onto \mathbb{R} will be also continuous.

Let $[t]$ be the largest integer n , $n \leq t$. Let us split $[0, \infty)$ into disjoint union of consecutive intervals I_k , $k \geq 0$ with $I_0 = [0, 1)$, $I_k = [3^{k-1}, 3^k)$. Note that $|I_k| = (2/3)3^k$. For $t > 0$ we define

$$W(t) = W_k(t - [t]) \text{ for } t \in I_k,$$

and extend it symmetrically $W(-t) = W(t)$ to the whole real line.

Clearly the weight W is not a matrix \mathbf{A}_2 weight (consider averages over intervals $[k, k+1]$ with $k \rightarrow \infty$). So, it remains to show that the scalar weights $(W(\cdot)x, x)$ and $(W^{-1}(\cdot)x, x)$ are Muckenhoupt weights. Note, that A_2 condition holds trivially if we restrict ourselves to intervals $[k, k+1]$, we need to extend this estimate to all intervals.

Lemma 5.3. *Let \tilde{u} be a Muckenhoupt A_2 weight on $[0, 1]$, i.e.*

$$\int_I \tilde{u} \cdot \int_I \tilde{u}^{-1} \leq C \quad \text{for all intervals } I \subset [0, 1]$$

Let

$$u(t) := \begin{cases} \tilde{u}(2t), & t \in [0, 1/2] \\ \tilde{u}(2t-1) & t \in (1/2, 1]. \end{cases}$$

Let w be the periodic extension of u

$$w(t) = u(t - [t]), \quad t \in \mathbb{R}.$$

Then w is an A_2 weight on \mathbb{R}

$$(5.3) \quad \int_I w \cdot \int_I w^{-1} \leq 4C \quad \text{for all intervals } I \subset \mathbb{R}.$$

Proof. The inequality (5.3) trivially holds for any I of form $[m/2, n/2]$, $m, n \in \mathbb{Z}$.

If $|I| \geq 1/2$, consider the smallest interval $J = [m/2, n/2]$, $m, n \in \mathbb{Z}$ containing I . Note that $|J| \leq 2|I|$, and so

$$\int_I w \leq \frac{|J|}{|I|} \int_J w \leq 2 \int_J w.$$

Combining it with the similar estimate for w^{-1} we get (5.3).

It remains to consider the case $|I| \leq 1/2$. In this case $I \subset I_1 \cup I_2$ where $I_{1,2}$ are consecutive intervals of form $[n/2, (n+1)/2]$. Without loss of generality we can assume that either $I_1 = [-1/2, 0]$, $I_2 = [0, 1/2]$ or $I_1 = [0, 1/2]$, $I_2 = [1/2, 1]$.

Consider the first case, $I_1 = [-1/2, 0]$, $I_2 = [0, 1/2]$. Let $J_{1,2} = I \cap I_{1,2}$. Assume for the definiteness that $|J_1| \geq |J_2|$. Then

$$\int_I w \leq \frac{1}{|I|} \int_{J_1 \cup (-J_1)} w = \frac{2}{|I|} \int_{J_1} w = \frac{2|J_1|}{|I|} \int_{J_1} w \leq 2 \int_{J_1} w;$$

the first equality in the chain holds because by the construction $w(t) = w(-t)$. Combining this estimate with the one for w^{-1} we get (5.3)

The case $I_1 = [0, 1/2]$, $I_2 = [1/2, 1]$ can be treated by shifting everything by $1/2$ to the left. It is easy to see that the function \tilde{w} , $\tilde{w}(t) = w(t + 1/2)$ we still symmetric, $\tilde{w}(t) = \tilde{w}(-t)$. \square

Let us note that the weights W_k satisfy the estimates

$$(5.4) \quad W_{k+m} \leq W_k \leq 2^m W_{k+m}, \quad k, m \geq 0,$$

and therefore

$$(5.5) \quad 2^{-m} W_{k+m}^{-1} \leq W_k^{-1} \leq W_{k+m}$$

To see that, one can notice that matrices $P + \varepsilon_k Q$ from which weights W_k were constructed, satisfy the inequalities, and all further operation do not change them.

Fix vector x . Inequalities (5.4) imply that the weights $w_k(\cdot) = w_{k,x}(\cdot) = (W_k(\cdot)x, x)$ satisfy

$$(5.6) \quad w_{k+m} \leq w_k \leq 2^m w_{k+m}, \quad k, m \geq 0.$$

Inequalities (5.5) imply similar estimates for the weight v_k^{-1} , where $v_k(\cdot) = (W_k^{-1}(\cdot)x, x)$.

We want to show that the weight $w(\cdot) = w_x(\cdot)$ satisfies the A_2 condition with the constant independent of x . Note, that it is sufficient to check the A_2 condition only for intervals $I \subset [0, \infty)$. Indeed, because $w(-t) = w(t)$

we can always replace $I \subset (-\infty, 0]$ by $-I$. As for the general interval I , the integrals over I are comparable with integrals over $I_+ := I \cup -I \cap [0, \infty)$,

$$\int_{I_+} w dt \leq \int_I w dt \leq 2 \int_{I_+} w dt.$$

So, let us assume without loss of generality that $I \subset [0, \infty)$.

We know that all $w_k = w_{x,k}$ satisfy A_2 condition with some constant C (independent of x and k). Lemma 5.3 implies that for an interval $I \subset I_k$

$$\int_I w \cdot \int_I w^{-1} \leq 4C.$$

If the interval I is contained in 2 consecutive intervals I_k , $I \subset I_k \cup I_{k+1}$, the weight w is equivalent on the interval $I_k \cup I_{k+1}$ to the periodic extension of the weight w_k , see (5.6) with $m = 1$. But by Lemma 5.3 the latter weight is A_2 with constant $4C$, so the above inequality holds (with constant $8C$) for $I \subset I_k \cup I_{k+1}$.

Let us now consider the last case, when $I \subset [0, \infty)$ intersects at least 3 of the I_k . Choose minimal k and m such that $I \subset \cup_{j=k-m}^k I_j$. It follows from (5.6) that $w(t)^{-1} \leq w_k(t - [t])^{-1}$ for $t \in I$ and so

$$(5.7) \quad \int_I w(t)^{-1} dt \leq \int_I w_k(t - [t])^{-1} dt \leq \frac{3}{2} \int_{[0,1]} w_k(t)^{-1} dt$$

To explain the last inequality, note that by periodicity of $w_k(t - [t])$

$$\int_J w_k(t - [t])^{-1} dt = \int_{[0,1]} w_k(t)^{-1} dt$$

for any interval J of integer length. Therefore we can continue estimates (5.7):

$$\int_I w_k(t - [t])^{-1} dt \leq \frac{|J|}{|I|} \int_J w_k(t - [t])^{-1} dt = \frac{|J|}{|I|} \int_{[0,1]} w_k(t)^{-1} dt,$$

where $J \subset I$ is an interval of integer length. But by the choice of k $I_{k-1} \subset I$, so $|I| \geq |I_{k-1}| = (2/3)3^{k-1} \geq 2$, so it is always possible to find an interval of $J \subset I$ integer length such that $|J|/|I| < 3/2$.

To estimate $\int_I w$ let us notice that $|I| \geq |I_{k-1}| = (2/3)3^{k-1}$. We now have

$$\begin{aligned} \int_I w(t) dt &\leq \frac{1}{|I|} \int_{3^{k-m-1}}^{3^k} w(t) dt \leq \sum_{j=k-m}^k \frac{|I_j|}{|I|} \int_{I_j} w_j(t - [t]) dt \\ &\leq \sum_{j=k-m}^k \frac{3}{2} 3^{-(k-j-1)} 2^{k-j} \int_{I_j} w_k(t - [t]) dt; \end{aligned}$$

the last inequality here follows from (5.6). Again the periodicity of $w_k(t - [t])$

$$\int_{I_j} w_k(t - [t]) dt = \int_{[0,1]} w_k(t) dt \quad \forall j.$$

So by summing the geometric series we get the bound

$$(5.8) \quad \frac{1}{|I|} \int_I w(t) dt \leq \frac{3^3}{2} \int_{[0,1]} w_k(t) dt \leq$$

By multiplying (5.7) and (5.8) we see that $w = w_x$ satisfies the A_2 condition uniformly in $x \neq 0$.

The same reasoning applied to v^{-1} gives the estimates on the A_2 bounds of v .

6. CONCLUDING REMARKS AND OPEN PROBLEMS

As we had shown in the paper, the uniform A_2 condition for the scalar weights $(W(\cdot)x, x)$ and $(W^{-1}(\cdot)x, x)$ implies matrix \mathbf{A}_2 condition in dimension 2 (2×2 matrices) and does not imply it in dimension 6. The situation in dimension 3, 4, or 5 is still unclear.

Another interesting question is whether in dimension 2 the uniform A_2 condition only for the weights $(W(\cdot)x, x)$ is sufficient for the \mathbf{A}_2 condition. Using the same reasoning as in Section 4.1 one can easily construct a counterexample for 4×4 weights. Namely, if P is a orthogonal projection of rank 3 in \mathbb{R}^4 , then the weight W on SO_4 given by

$$W(U) = U^* P U, \quad U \in SO_4$$

clearly does not satisfy the matrix \mathbf{A}_2 condition, but using an obvious modification of the reasoning from Section (4.1) one can show the scalar weights $(W(\cdot)x, x)$ are uniformly A_2 . So far we were not able to get a counterexample in lower dimensions, although we suspect it should exist even in dimension 2.

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