Estimates in the Operator Corona Problem.

Sergei Treil
Brown University

Given an operator-valued function $F \in H^\infty(E \to E_\ast)$, the operator corona problem consists of finding criteria for the existence of an analytic left inverse $G \in H^\infty(E_\ast \to E)$, $GF \equiv I$. A simple necessary condition is

$$F^*(z)F(z) \geq \delta^2 I, \quad \forall z \in D \quad (\delta > 0).$$

If this condition is sufficient, we say that the Operator Corona Theorem holds. It is known that if $\dim E < \infty$, this condition is sufficient: for $\dim E = 1$, $\dim E_\ast < \infty$ this is a famous Carleson Corona Theorem; the case $\dim E = 1$, $\dim E_\ast = \infty$ was proved independently by M. Rosenblum, V. Tolokonnikov and A. Uchiyama by a modification of the Wolff’s proof of the Corona Theorem. Then the case $\dim E_\ast < \infty$ was obtained from the case $\dim E = 1$ via simple linear algebra arguments independently by V. Vasyunin and P. Fuhrmann. However in the general case the Operator Corona Theorem fails: it was shown by the author that the norm of the solution $G$ can grow as $\log^{1/2}(\dim E)$ (for a fixed $\delta$).

The known upper bounds for the norm are exponential, $C^{\dim E}$.

In the talks we will discuss new estimates in the Operator Corona Theorem.

In the first talk, we will show that there exists an $(n + 1) \times n$ matrix function $F \in H^\infty$ satisfying $I \geq F^*F \geq \delta^2 I \ (0 < \delta < 1/4)$ and such that the norm of its left inverse in $H^\infty$ is at least $C \cdot \left(\frac{\delta}{1-\delta}\right)^{-n}$. This shows that the correct bounds are indeed exponential in $\dim E$, and also gives a counterexample to the so-called “codimension one conjecture.” Save for a few well known facts like the commutant lifting theorem, the proofs are quite elementary.

In the second talk we discuss some results in positive direction. While the general Operator Corona Theorem fails, it holds for perturbations of “nice” operator-functions. For example, it holds if we perturb a left invertible in $H^\infty$ operator-function by a function in $H^\infty(\mathcal{S}_2)$ ($H^\infty$ with values in the Hilbert–Schmidt operators). The technique used to prove such results involves solving $\overline{\partial}$-equations on holomorphic vector bundles. While people familiar with complex differential geometry will definitely recognize the ideas, I’ll try to make the proof understandable to analysts. In fact the main tool I’ll be using is the Green’s formula.