Margulis constant

Jørgensen Inequality

\[ \langle f, g \rangle \text{ non-deg } f, g \in \text{PSL}_2 \mathbb{C} \]

\[ \mu(f,g) = |\text{tr}^2 f - 4| + |\text{tr} [f,g] - 2| \geq 1 \]

Lemma: \( f \in \text{PSL}_2 \mathbb{C} \)

\[ \Theta_f : \text{PSL}_2 \mathbb{C} \to \text{PSL}_2 \mathbb{C} \quad \Theta_f(g) = gf g^{-1} \]

If \( \langle f, g \rangle \) klein, satisfies \( \Theta^n_f(g) = f \) for some \( n \), and \( \text{ord}(f) \neq 2 \), then \( \text{Fix}(f) \) is invariant under \( g \). In particular, \( \langle f, g \rangle \) is elem.

pf: Set \( \text{Fix}(f) = \{x, y \} \). \( g_m = \Theta^n_f(g) \) not of order 2. \( \text{Fix}(g_m) = \{g_m(x), g_m(y)\} = \{x, y \} \) then \( \text{Fix}(g_{m+1}) = \{x, y \} \).

This implies \( \text{Fix}(g_m) = \{x, y \} \), b/c \( g_m \) is not of order 2.

Inductively \( \text{Fix}(g_i) = \{x, y \} \) and \( \{x, y \} \) is invariant under \( g \), so invariant under \( \langle f, g \rangle \), so elementary.

Proof of Jørgensen Inequality: Define \( \Theta_f \) as before. WTS \( \exists n \Theta^n_f(g) = f \) if \( \mu = \mu(f,g) < 1 \), giving a contradiction.

Suppose \( f \) loxod. or elliptic. \( f \) is not of order 2, since \( \text{tr}^2 f = 0 \)

So \( \mu \geq 4 \). Set

\[
\begin{align*}
    f &= \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix} \\
    g &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
    \Theta^n_f(g) &= g_n = \begin{pmatrix} a^n & b^n \\ c^n & d^n \end{pmatrix}
\end{align*}
\]

\[ \text{tr} f = u + \frac{1}{u} \]

\[ [f, g] = f g f^{-1} g^{-1} = \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix} \begin{pmatrix} d & -b \\ c & a \end{pmatrix} \]

\[
\begin{pmatrix}
    u a & u b \\
    c & d
\end{pmatrix} \begin{pmatrix}
    u & 0 \\
    -c & a
\end{pmatrix}
\begin{pmatrix}
    d & -b \\
    c & a
\end{pmatrix}
\begin{pmatrix}
    d & -b \\
    c & a
\end{pmatrix}
\begin{pmatrix}
    u a & u b \\
    c & d
\end{pmatrix}
\]

\[
\begin{pmatrix}
    (ad-bc)u^2 & -ab + a^2 u^2 \\
    cd/u^2 & -cd - cd/u^2 + ad
\end{pmatrix}
\]
\[
\text{tr} [fg] = 2ad - bc u^2 - cb/u^2
\]
\[
|\text{tr}^2 f - 4| = |(u + \frac{1}{u})^2 - 4|
\]
\[
= |u^2 + 2 + \frac{1}{u^2} - 4|
\]
\[
= |u^2 + \frac{1}{u^2} - 2|
\]
\[
= |u - u'|^2
\]
\[
|\text{tr} [f, g] - 2| = |2ad - 2bc u^2 - cb/u^2|
\]
\[
= |2bc - bc u^2 - cb/u^2| \quad \text{ad} - bc = 1
\]
\[
= |bc| |u - u'|^2 \quad 2ad - 2bc = 2 \quad 2ad - 2 = 2bc
\]

so \(\mu(f, g) = |bc + 1| |u - u'|^2\)

Since \(g_{n+1} = g_n f g_n^{-1}\)

\[
\begin{pmatrix}
  a_{n+1} & b_{n+1} \\
  c_{n+1} & d_{n+1}
\end{pmatrix} = \begin{pmatrix}
  a_n & b_n \\
  c_n & d_n
\end{pmatrix}\begin{pmatrix}
  u & 0 \\
  0 & u'
\end{pmatrix}\begin{pmatrix}
  d_n & -b_n \\
  -c_n & a_n
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  a_n u & b_n u^{-1} \\
  c_n u & d_n u'
\end{pmatrix}\begin{pmatrix}
  d_n & -b_n \\
  -c_n & a_n
\end{pmatrix} = \begin{pmatrix}
  a_n d_n u - b_n c_n u^{-1} & -a_n b_n u \\
  c_n d_n u^{-1} - b_n c_n u & a_n b_n u^{-1}
\end{pmatrix}
\]

thus \(b_{n+1}, c_{n+1} = a_n b_n (u - u') c_n d_n (u - u')\)
\[(u-u')^2 \quad \text{and} \quad n = 1\]

Thus, inductively
\[\|bnC_n\| \leq \mu^n \|bc\|\]

So if \(\mu < 1\) \(\text{and} \quad n = 1 + bnC_n \rightarrow 1\)

Thus
\[g_{n+1} = and_n u - bnC_n u' \rightarrow u\]
\[d_{n+1} = and_n u' - bnC_n u \rightarrow u'\]

\[\frac{\|bn+1\|}{\|bn\|} = \|an(u'-u)\| \leq \|u\| \|u' - u\| \leq \|u\| \frac{\mu}{\sqrt{2}}\]

\[\frac{\|cn+1\|}{\|cn\|} = \|dn(u-u')\| \leq \|u'\| \|u - u'\| \leq \|u'\| \mu^2\]

Thus, \(g_n\) converges, and by discreteness of \(\langle f, g \rangle\)

is constant for large \(n\). So
\[g_n = g_{n+1} = g_{n+2} = \ldots \]

for some \(n\), thus
\[g_n = f\].

Next, suppose \(f\) parabolic
\[f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad |\text{tr}^2 f - 4| = 0\]

\[[fg] = fgf^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -a & -b \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} \begin{pmatrix} d+c & b-a \\ -c & a \end{pmatrix}\]
\[
\begin{align*}
&\left( (a+c)(d+c) - c(b+d) \right) - (a+c)(b+a) + a(b+d) \\
&\left( c(d+c) + d(d+c) \right) - (c(b+a) + ad)
\end{align*}
\]

\[
\text{tr } [f,g] = \begin{align*}
ad + ac + cd + c^2 - cb - cd - ca + ad
\end{align*}
\]

\[
= 2 + c^2
\]

\[
\text{ad - bc} = 1
\]

\[
1 \text{tr } [f,g] - 2 = |c|^2
\]

So \( \mu = |c|^2 \). Moreover,

\[
\begin{pmatrix}
an & bn \\
cn & dn
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
dn & -bn \\
-cn & an
\end{pmatrix}
\]

\[
= \begin{pmatrix}
an & an + bn \\
cn & cn + dn
\end{pmatrix}
\begin{pmatrix}
dn & -bn \\
-cn & an
\end{pmatrix}
\]

\[
= \begin{pmatrix}
andn - ancn - bncn & -anbn + \alpha_n^2 + anbn \\
-cnbn + an(cn + an cn) & cndn - Cndn - C_n^2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
l - an cn & \alpha_n^2 \\
-cn^2 & 1 + an(cn)
\end{pmatrix}
\]
Observe that \( c_{n+1} = -c_n^2 = -(c_{n-1})^2 = -c_{n-1}^4 \), etc.

It is then clear that \( \mu = |c|^2 < 1 \) implies \( c_n \to 0 \)

Since \( a_{n+1} = 1 - an cn \)

\[
|a_{n+1}| = |1 - cn an| \\ 
\leq |1 - 2an| \\ 
\leq 1 + \frac{1}{2}|an| \quad \text{for } n \gg 0 \text{ s.t. } |cn| \leq \frac{1}{2}
\]

Thus \( a_n \) is bounded so \( an cn \to 0 \)

so \( an \to 1 \)

Hence \( b_{n+1} = a_n^2 \) implies \( bn \to 1 \)

\( d_{n+1} = 1 + an cn \) implies \( ch \to 1 \)

Thus the \( q_n \) are the same for sufficiently large \( n \), and \( q_n : f \) for some \( n \), so \( \langle f, g \rangle \) is elementary.

(We also know \( g \) fixes \( \text{Fix}(f) = \infty \), so \( c = 0 \))

Corollary: If \( \langle f, g \rangle \) Kleinian satisfies \( \mu(f, g) < 1 \), then \( \text{Fix}(f) \) is \( g \)-invariant.

Lemma: Let \( \langle f_n, g_n \rangle \) be non-elem. Kleinian, \( f_n, g_n \to f, g \) in \( \text{PSL}_2 \mathbb{C} \).

Then

1. \( f \neq 1 \)
2. If \( \{f_n\} \) contains no elliptics, \( f \) is not elliptic.
proof: If \( f_n \to id \) then \( \mu(f_n, g_n) \to id \) as \( n \to \infty \).
Thus \( \langle f_n, g_n \rangle \) is elementary for \( n \) sufficiently large, giving a contradiction.

(2) If \( f \) is elliptic of order \( m \), \( f^m = id \) and so apply part (1).
If \( f \) is \( \infty \)-order, take \( f^k \) very close to the identity, so \( \mu(f^k, g) < 1 \). We have

\[ \mu(f^k, g) \to \mu(f^k, g) \]

So \( \text{Fix}(f^k) \) is \( g \)-invariant for \( n \) suff. large. Then since \( \text{Fix}(f^k) = \text{Fix}(f) \), \( \langle f_n, g_n \rangle \) must be elementary, another contradiction.

Theorem: \( \Gamma \) non-elem. Kleinian groups, \( \rho_n: \Gamma_0 \to \Gamma_n \).
Further suppose \( \rho_n(\gamma) \) converges in \( \text{PSL}_2 \mathbb{C} \) \( \forall \gamma \in \gamma_0 \), to some limit \( \rho(\gamma) \). Then

\[ \Gamma = \{ \rho(\gamma) \} \gamma \in \gamma_0 \]

is a non-elem. Klein group and \( \rho: \Gamma_0 \to \Gamma \).

pf: It is easy to see that \( \Gamma \) is a group and \( \rho \) a homom. E.g.

\[ \rho(\gamma), \rho(\omega) \in \Gamma_0 \text{ then } \rho(\gamma) \rho(\omega) \in \Gamma_0 \text{ b/c} \]

\[ \rho_n(\gamma \omega) = \rho_n(\gamma) \rho_n(\omega) \to \rho(\gamma) \rho(\omega) \in \Gamma_0, \text{ etc.} \]

Let \( H_0 = \text{Ker} \rho \). Suppose \( H_0 \) is non-trivial.

Claim: \( H \subset \Gamma \), \( \Gamma \) Klein non-elem, \( H \) normal, then

\[ \Lambda(H) = \Lambda(\Gamma) \text{ if } H \neq \{ id \}. \]

pf: Since \( \gamma H \gamma^{-1} = H \forall \gamma \in \Gamma \), \( \Lambda(H) \) is \( \Gamma \)-invariant.
Suppose \( \Lambda(H) \) non-empty. Since it is closed, \( \Lambda(H) \cap \Lambda(\Gamma) \).
\[ \Lambda(H) \cap \Lambda(\Gamma) \text{ is obvious, so } \Lambda(H) = \Lambda(\Gamma). \]
If \( \Lambda(H) \) is empty, \( H \) is finite. Then \( U(\chi^h) \) is finite, for \( h \in H \cdot \{ \text{id} \} \). This is also \( \Gamma \) invariant by normality, implying \( \Gamma \) elementary unless \( H = \{ \text{id} \} \).

Thus \( H_0 \) is non-elementary. Choose a free rank 2 subgroup \( \langle f_0, g_0 \rangle \) in \( H_0 \). Let \( f_n = \rho_n(f_0) \), \( g_n = \rho_n(g_0) \), and \( H_n = \langle f_n, g_n \rangle \).

Since \( \rho_n \) is an isomorphism, \( H_n \) is non-elementary. Thus \( \rho(f_0) \) is not the identity by our lemma, and hence cannot be in the kernel, so \( H_0 \) is trivial.

Lastly, we show \( \Gamma \) discrete. Take \( y_1, y_2 \in \Gamma \) s.t.

\[
\langle \rho^{-1}(y_1), \rho^{-1}(y_2) \rangle
\]

is a free subgroup of \( \Gamma_0 \).

If \( \Gamma \) is not discrete, there exists \( h \in \Gamma \cdot \{ \text{id} \} \) sufficiently close to the identity. Then for \( n \gg 0 \)

\[
\mu(\rho_n \circ \rho^{-1}(h), \rho_n \circ \rho^{-1}(y_j)) < 1 \quad (j = 1, 2)
\]

So \( \text{Fix}(\rho_n \circ \rho^{-1}(h)) \) is invariant under \( \rho_n \circ \rho^{-1}(y_j) \).

So \( \langle \rho_n \circ \rho^{-1}(y_1), \rho_n \circ \rho^{-1}(y_2) \rangle \) is elementary, preserving \( \text{Fix}(\rho_n \circ \rho^{-1}(h)) \). But then \( \langle \rho^{-1}(y_1), \rho^{-1}(y_2) \rangle \) would be elementary, giving a contradiction.

(Rmk: By argument similar to these, one can show \( \Gamma < PSL_2(\mathbb{C}) \) is discrete iff any two-generator subgroup of \( \Gamma \) is (Jørgensen 1976).)
Let $U_r(p)$ be a hyperbolic open ball in $\mathbb{H}^3$ with center $p$ and radius $r$. Let $I(\Gamma; p, r) = \{ \gamma \in \Gamma \mid \gamma(U_r(p)) \cap U_r(p) \neq \emptyset \}$ and $\Gamma(p, r) = \langle I(\Gamma; p, r) \rangle$.

**Margulis Lemma.** \( \exists r_0 > 0 \) s.t. \( \forall p \in \mathbb{H}^3 \), \( \Gamma \) torsion-free Kleinian \( \Gamma(p, r) \) is elementary for \( r \leq r_0 \).

**pf:** Let $I(\Gamma; p, r) = \{ h_1, \ldots, h_m \}$. To show $\Gamma(p, r)$ is elementary, we'll show $\langle h_i, h_k \rangle$ is elementary for all $i, k$. This implies $\text{Fix}(h_i) = \text{Fix}(h_k) = N(\Gamma)$.

Assume $p \in \mathbb{H}^3$ is $(0, 0, 1)$ and no such constant exists. Choose $\Gamma_n$ and $f_n, g_n$ in $I(\Gamma; p, \frac{1}{n})$ with $\langle f_n, g_n \rangle$ non-elementary. We know

$$\|f_n\|^2 = 2 \cosh d(p, f_n(p)) < 2 \cosh \frac{1}{n}$$

Similarly for $\|g_n\|^2$. Passing to a subsequence if necessary,

$$f_n, g_n \to f, g \text{ in } \text{PSL}_2 \mathbb{C}. f \text{ fixes } p, \text{ so is elliptic or } \text{id}, \text{ but this contradicts the prior lemma.} \quad \Box$$

**Def:** $N = N_\Gamma$ complete $\mathbb{H}^3$ m, $\varepsilon > 0$, $N_{(0, \varepsilon)}$ consists of points $p \in N_\Gamma$ through pass a nontrivial closed curve of length $2\varepsilon$.

Write $N_{(0, \varepsilon)} = N_{(\varepsilon, \infty)}$, $N_{(0, \varepsilon)} + N_{\text{thin}}, N_{(\varepsilon, \infty)} = N_{\text{thick}}$

- $N_{(0, \varepsilon)}$ is where injectivity radius is $< \varepsilon/2$.

**Thm:** $N_\Gamma$ complete $\mathbb{H}^3$ m, $\varepsilon \leq 2r_0$, then each connected component of $N_{(0, \varepsilon)}$ is isomorphic to one of 3 types.

For $\varepsilon > 0$ define $c = (c(\varepsilon))$ s.t. $d((0, 0, c), (1, 0, 0)) = \varepsilon$. Let

$$H_c = \{ p = (x, y, t) \in \mathbb{H}^3 \mid t > c^2 \}$$
(1) \( \mathbb{H}/J_1 \): \( J_1 = \langle z \mapsto z+1 \rangle \) cyclic parabolic (cusp cylinder)

(2) \( \mathbb{H}/J_2 \): \( J_2 = \langle z \mapsto z+1, z \mapsto z + \bar{z} \rangle \) parabolic (cusp tube)
\[ \text{Im } \bar{z} > 0 \quad |\bar{z}| \geq 1 \]
\( \text{abelian} \), rank 2

(3) \( U/\langle \gamma \rangle \): solid torus, \( \gamma \) loxodromic element (Margulis solid)
\( U \) tubular nbhd of axis \( A_\gamma \) (torus)

pf: For \( \gamma \in \Gamma - \{ \text{id} \} \), set
\[ \tilde{\gamma}_\gamma = \{ p \in \mathbb{H}^3 \mid d(\gamma, \gamma(p)) < \varepsilon \} \]

If \( \gamma(z) = z+1 \), by def. of \( \gamma \) we have \( \tilde{\gamma}_\gamma = \mathbb{H} \). If \( \gamma \) loxodromic and \( \tilde{\gamma}_\gamma \neq \emptyset \), it is a tubular nbhd of \( A_\gamma \), and to see this consider \( \gamma = 2z \).

Let \( \text{pr} : \mathbb{H}^3 \rightarrow N \), and observe
\[ \text{pr}^{-1}(N_{(0,\varepsilon)}) = \bigcup_{\gamma \in \Gamma - \{ \text{id} \}} \tilde{\gamma}_\gamma \]

Take connected component \( \tilde{E} \) of \( \text{pr}^{-1}(N_{(0,\varepsilon)}) \)
Set \( \tilde{J} = \text{Stab}_\Gamma(\tilde{E}) \). Since \( \tilde{E} \) is precisely \( \tilde{J} \)-invariant
\[ \gamma \in \Gamma - \{ \text{id} \} \text{ is in } \tilde{J} \text{ iff } \tilde{\gamma}_\gamma \subset \tilde{E} \] Thus
\[ \tilde{E} = \bigcup_{\gamma \in \tilde{J} - \{ \text{id} \}} \tilde{\gamma}_\gamma \]

We prove \( \tilde{J} \) elementary. If \( \tilde{\gamma}_\gamma \cap \tilde{\gamma}_{\gamma'} \neq \emptyset \) for \( \gamma, \gamma' \in \tilde{J} - \{ \text{id} \} \), \( \langle \gamma, \gamma' \rangle \) is elementary by Margulis Lemma. Thus
\[ \text{Fix}(\gamma) = \text{Fix}(\gamma') \] Since \( \tilde{E} \) is connected
all nontrivial \( \gamma \in \tilde{J} \) have the same Fix point set.
So \( \tilde{J} \) is elementary.

The theorem follows by consider the possibilities for elementary \( J \).